Simplicity meets complexity

Reproducing complex connectivity patterns with simple models

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Outline

1. Are simple models enough to study complex systems/networks?

2. “Simple” ways to encode structural complexity

   (a) latent metric space

   (b) stub types



Louis J. Dubé



M. Ángeles Serrano



Marián Boguñá



Laurent Hébert-Dufresne

Collaborators














…and several others.

Postdocs, students and interns






















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1. **Are simple models enough to study complex systems/networks?**

2. “Simple” ways to encode structural complexity

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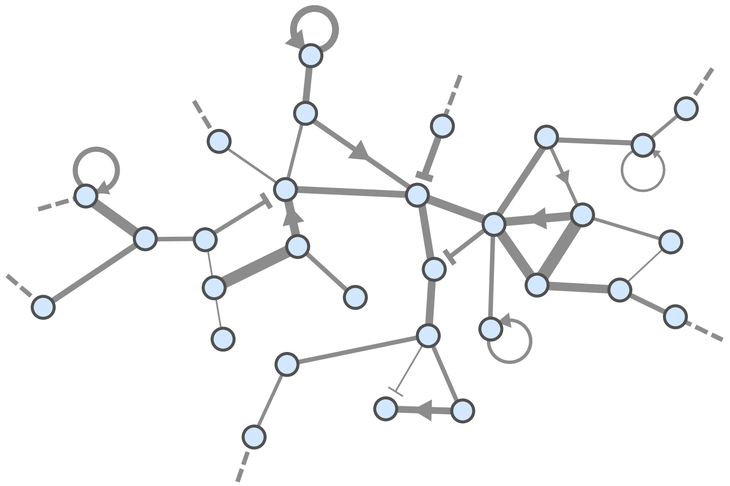
   (b) stub types



Modeling the dynamics of complex systems

Complex network

W

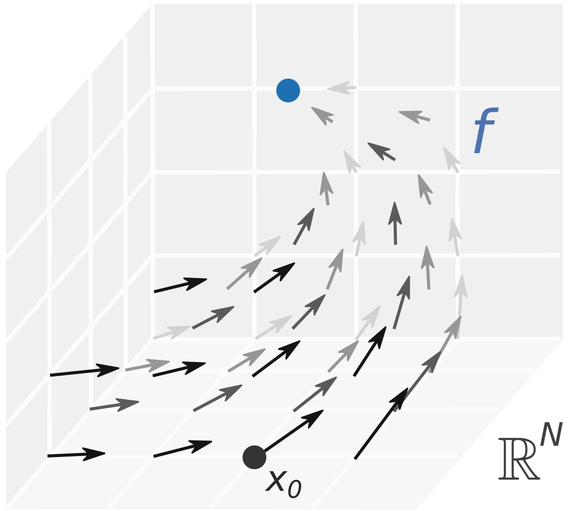


Vector field

f

High-dimensional dynamics

$\dot{x}=f(x;W)$



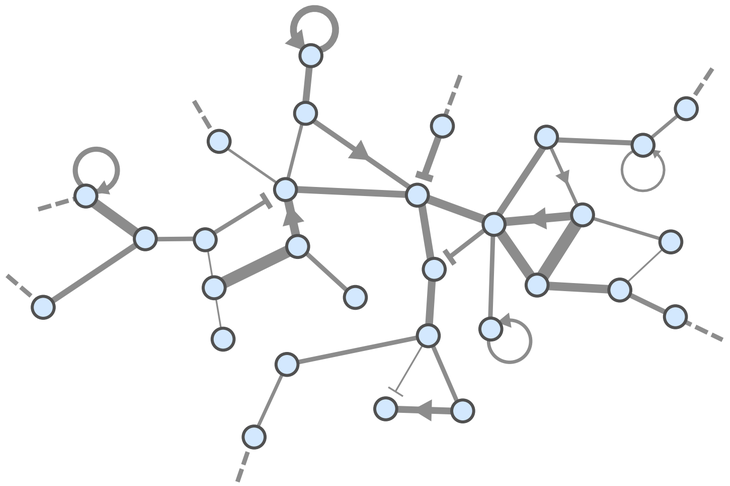
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Most mathematical models involve low-dimensional representations of the networks and/or dynamical observables.

Modeling the dynamics of complex systems

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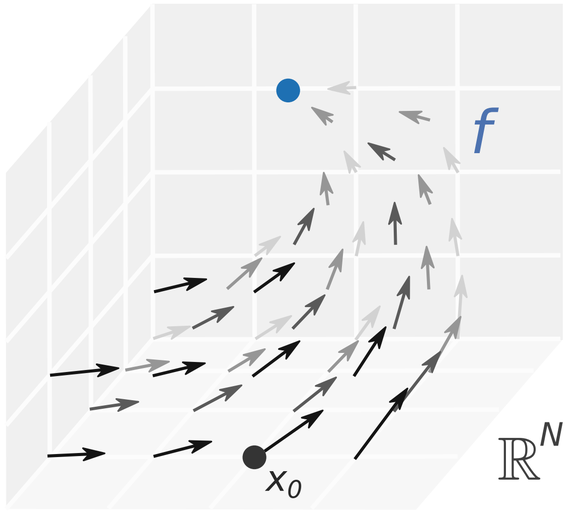


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Can we justify such a low-dimensional approach beyond mathematical and/or conceptual convenience?

The rank of adjacency matrices

Rank r of a matrix

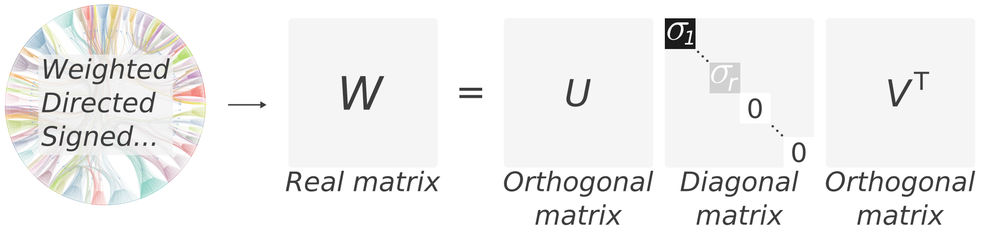
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Singular value decomposition (SVD)

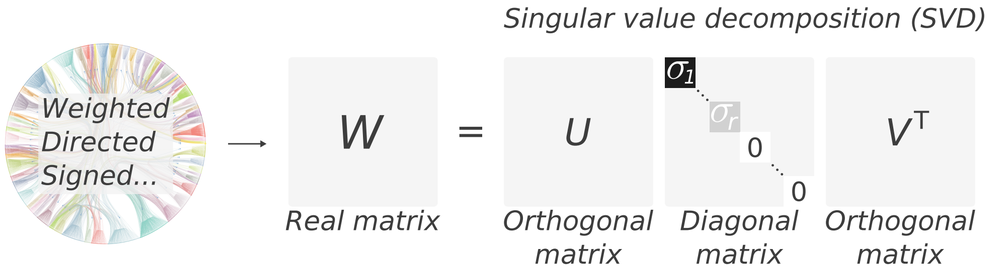


$$\mathbf{W} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\intercal = \sigma_1 \begin{bmatrix} | \\ \mathbf{u}_1 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_1^\intercal & — \end{bmatrix} + \sigma_2 \begin{bmatrix} | \\ \mathbf{u}_2 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_2^\intercal & — \end{bmatrix} + \sigma_3 \begin{bmatrix} | \\ \mathbf{u}_3 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_3^\intercal & — \end{bmatrix} + \dots$$

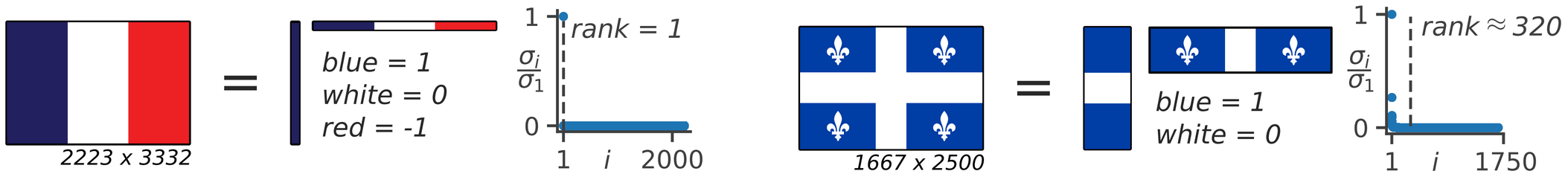
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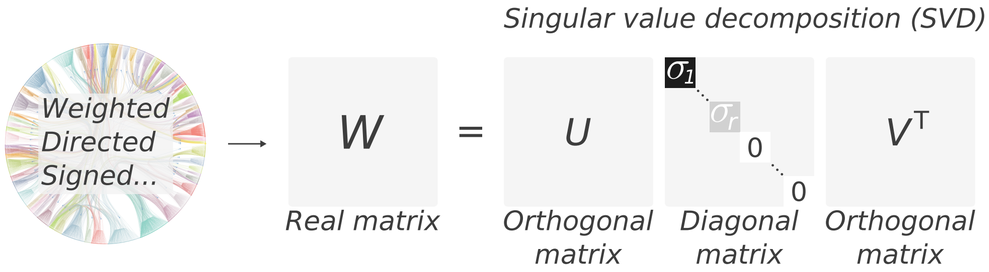
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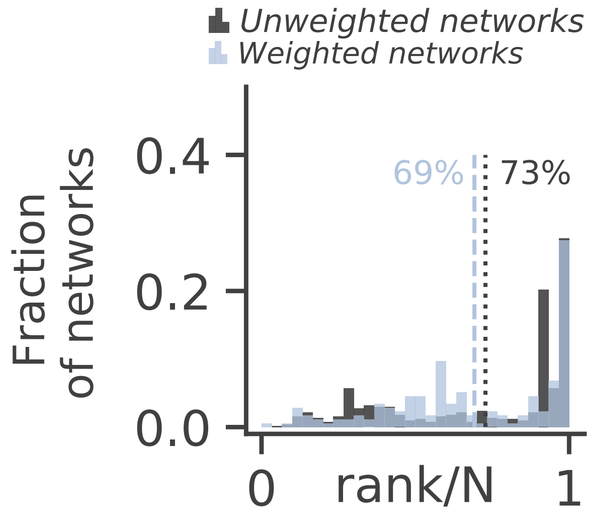
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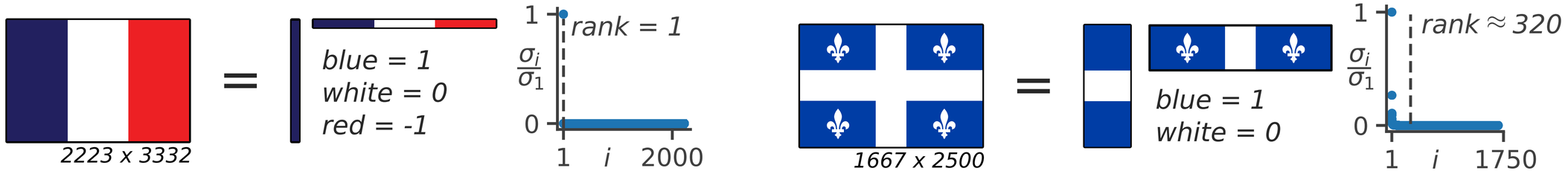


Full-rank (or almost full rank) adjacency matrices are very common!



Results for 679 empirical networks (502 unweighted networks and 177 weighted networks) downloaded from Netzschleuder.

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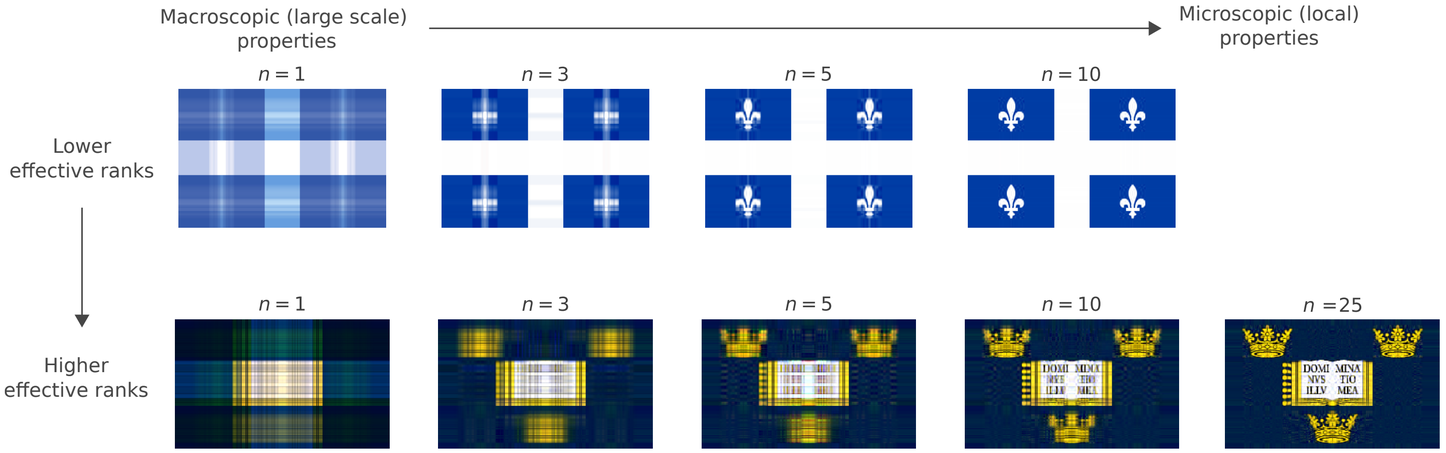


The effective ranks of adjacency matrices

$$\mathbf{W} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top \simeq \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$$

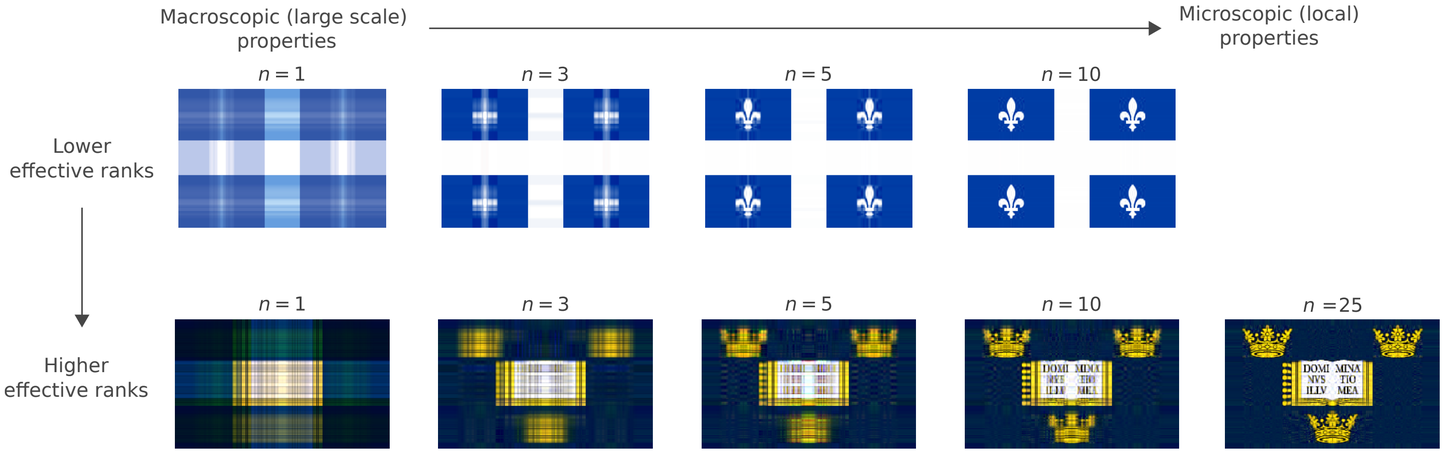
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Effective rank n of a matrix

- number of "significant" nonzero singular values
- term after which it is "reasonable" to truncate the sum

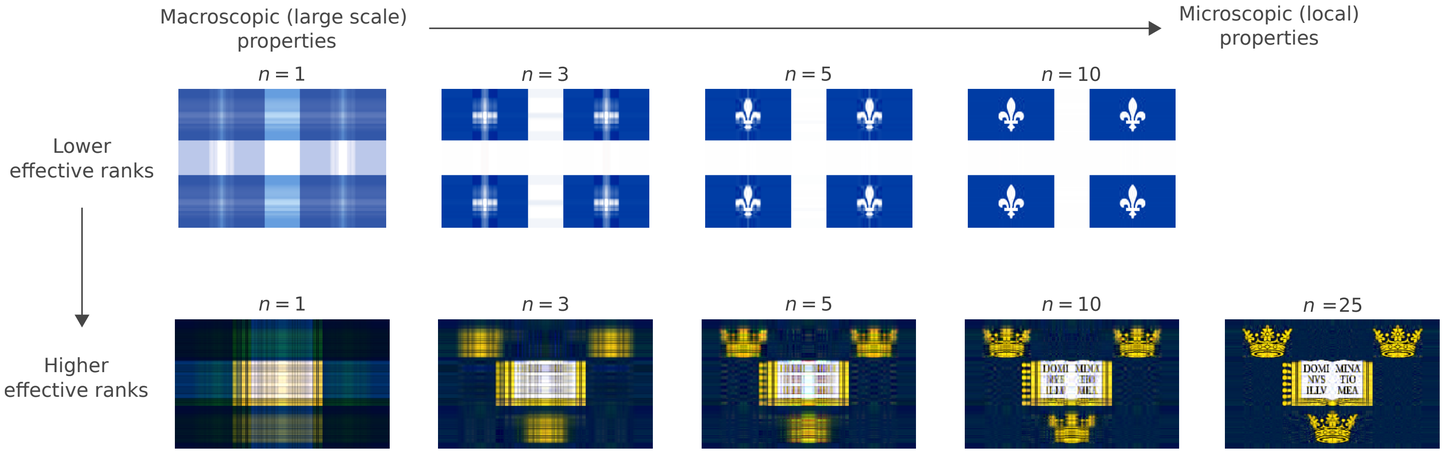
Abbreviation	Expression
srank	$\sum_{i=1}^{r} \sigma_i^2 / \sigma_1^2$
nrank	$\sum_{i=1}^{r} \sigma_i / \sigma_1$
energy	$\min\left[\arg\max_{\ell \in \{1,\ldots,N\}} \left(\sum_{i=1}^{\ell} \sigma_i^2 / \sum_{j=1}^{r} \sigma_j^2 > \tau\right)\right]$
elbow	$\frac{1}{\sqrt{2}} \arg\max_{i \in \{1,\ldots,N\}} \left\lvert \frac{i-1}{N-1} + \frac{\sigma_i - \sigma_N}{\sigma_1 - \sigma_N} - 1 \right\rvert - 1$
erank	$\exp\left\lvert -\sum_{i=1}^{r} \frac{\sigma_i}{\sum_{j=1}^{r} \sigma_j} \log \frac{\sigma_i}{\sum_{j=1}^{r} \sigma_j} \right\rvert$
thrank	$\#\left\{\sigma_i \mid i \in \{1, \ldots, N\} \text{ and } \sigma_i > \frac{4\sigma_{\text{med}}}{\sqrt{3}\mu_{\text{med}}}\right\}$
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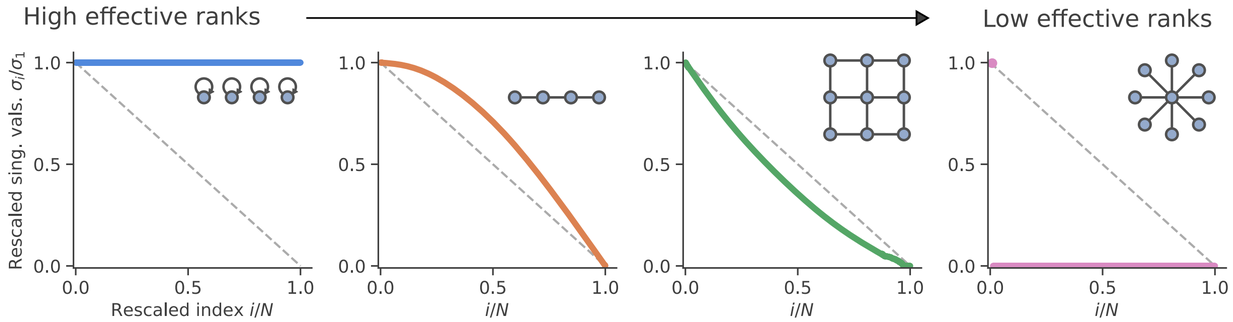
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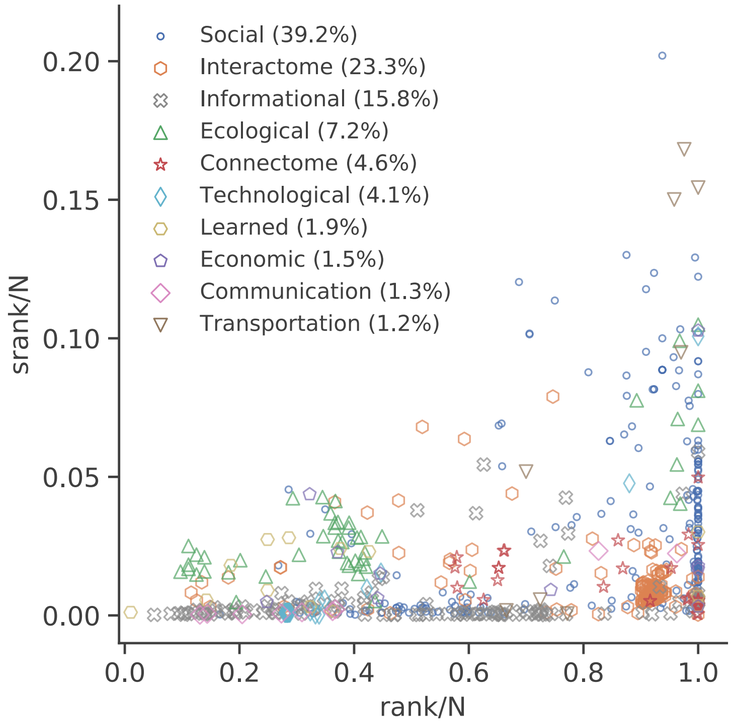
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The singular values (and the effective rank) encode information about the network's topology.



The effective ranks of adjacency matrices

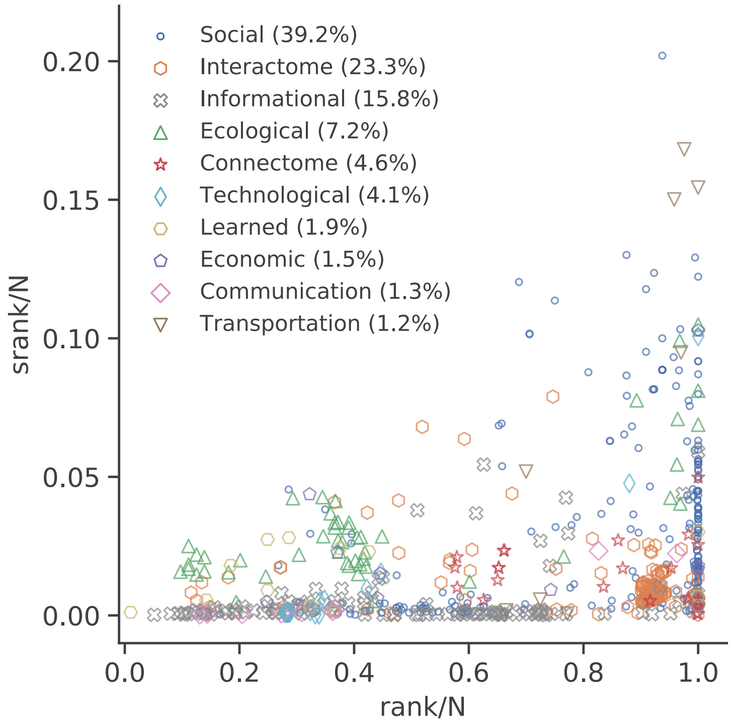
Many empirical networks appear to have a low effective rank!



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But what does “low” mean?

A *workable* definition of "low" effective rank

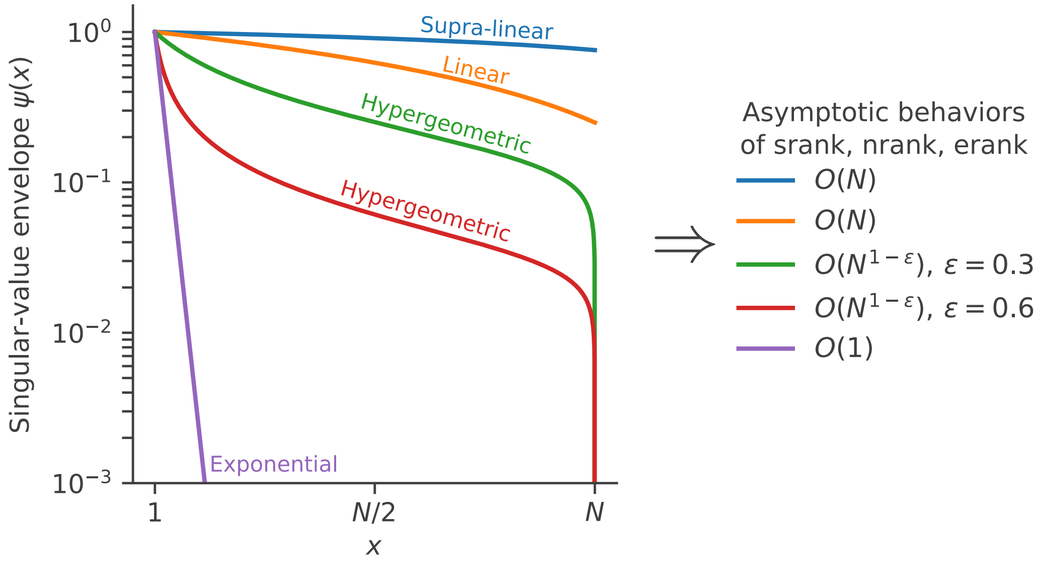
Hint: the rapid decrease of the dominant singular values of the adjacency matrix implies a low effective rank

- low effective rank? $\Rightarrow$ effective rank scales at most sublinearly as the number of nodes, N, goes to infitnity ($N^{1-\varepsilon}$ with $\varepsilon \in (0, 1]$)

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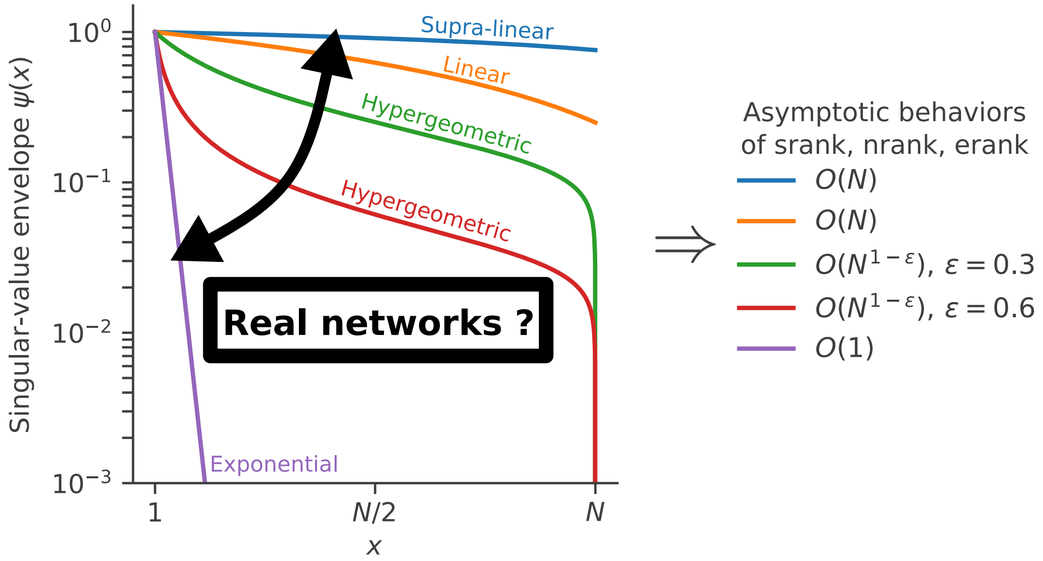
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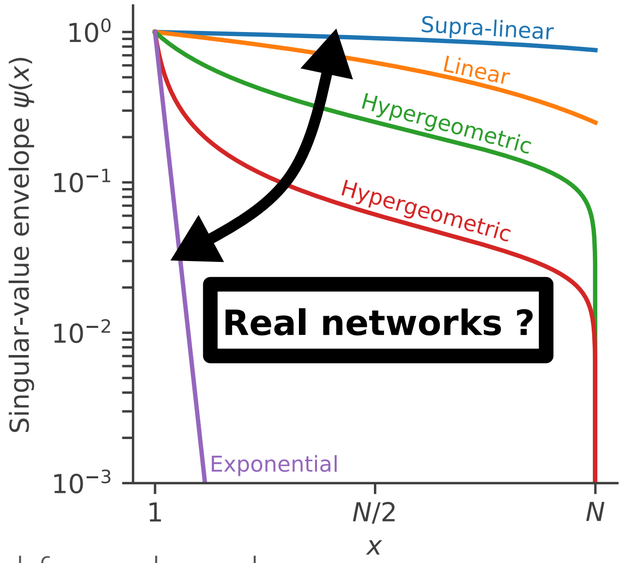
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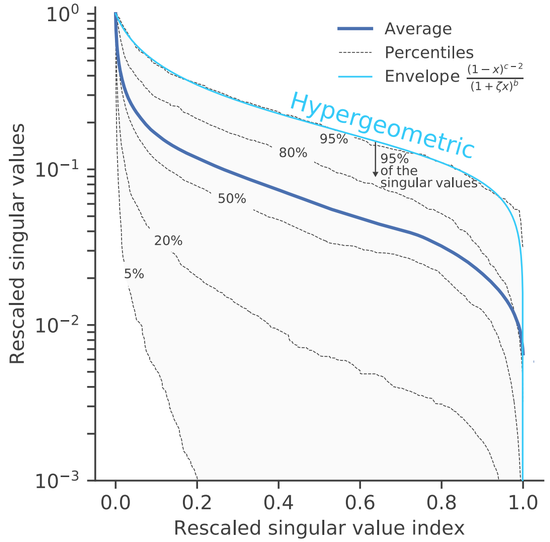


⇒

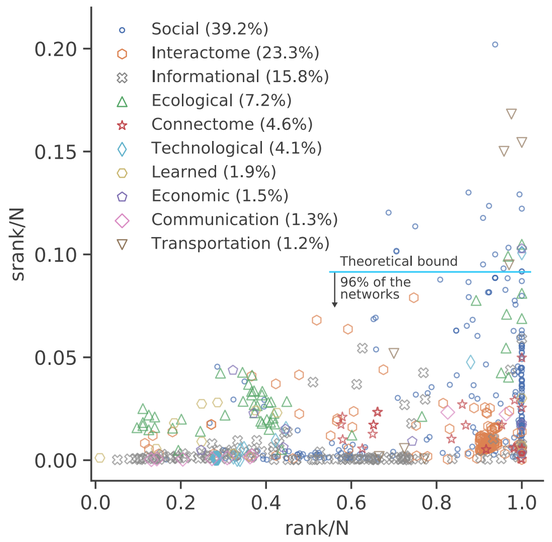
Asymptotic behaviors of srank, nrank, erank

- $O(N)$
- $O(N)$
- $O(N^{1-\varepsilon})$, $\varepsilon = 0.3$
- $O(N^{1-\varepsilon})$, $\varepsilon = 0.6$
- $O(1)$

The singular values of approx. 96% of considered networks are bounded from above by an hypergeometric envelope ⇒ sublinear effective ranks!



⇒



Workable definition of low effective rank: ∼10% of the number of nodes N

Approx. 96% of the 679 networks qualify for having a low effective rank!

The impact of effective low rank on the dynamics

Many dynamics on networks have the form

$$\dot{\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$$

with $\boldsymbol{x} \in \mathbb{R}^N$.

Examples:

- SIS (mean-field) : $\dot{x}_i = -d_i x_i + \gamma(1 - x_i)\, y_i$
- Wilson-Cowan: $\dot{x}_i = -d_i x_i + (1 - a x_i) \frac{1}{1+e^{-b(\gamma\, y_i - c)}}$
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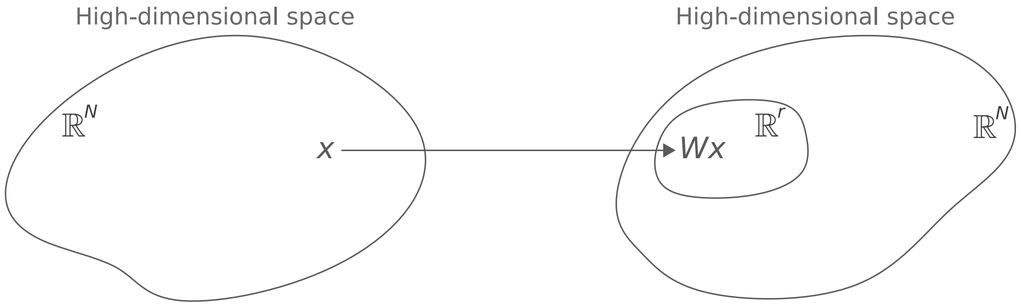
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Low rank $\mathbf{W} \Rightarrow \mathbf{W}\boldsymbol{x}$ belongs to a low-dimensional subspace



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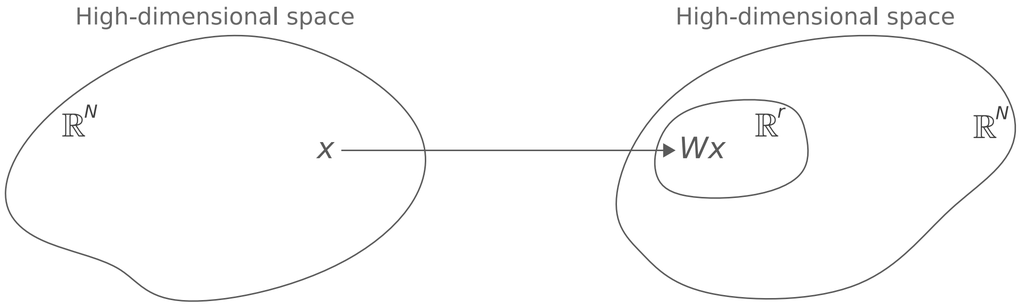
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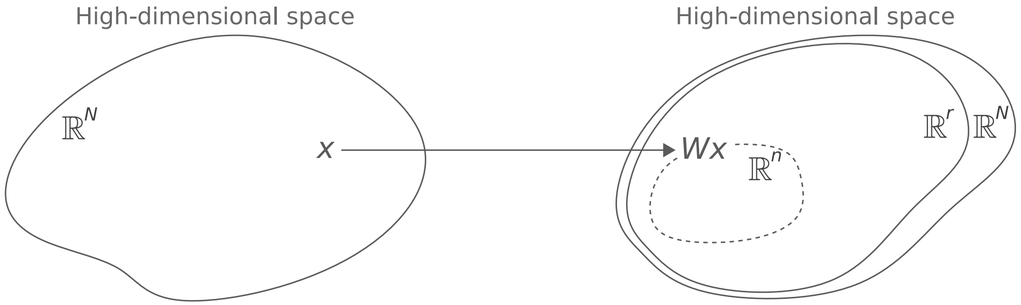
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Low *effective* rank $\mathbf{W} \Rightarrow \mathbf{W}\boldsymbol{x}$ belongs to an *effectively* low-dimensional subspace



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Original dynamics

$$\dot{\boldsymbol{x}} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x})$$

Reduced dynamics (with $\boldsymbol{X} = \mathbf{M}\boldsymbol{x}$)

$$\dot{\boldsymbol{X}} = \mathbf{M}\mathbf{g}(\mathbf{M}^{+}\mathbf{X}, \mathbf{W}\mathbf{M}^{+}\mathbf{X})$$

where n is the dimension of the reduced system
$\mathbf{M}^{+}$ denotes the pseudoinverse of $\mathbf{M}$
$\mathbf{M} = \mathbf{V}_n^{\top}$ is n-truncated right singular vector matrix

The alignment error is

$$\mathcal{E}(x) \leq \frac{1}{\sqrt{n}} \left[\|\mathbf{V}_n^{\top} \mathbf{J}'_x (\mathbf{I} - \mathbf{V}_n \mathbf{V}_n^{\top}) x\| + \sigma_{n+1} \|\mathbf{V}_n^{\top} \mathbf{J}'_y\|_2 \|x\| \right]$$

where $\mathbf{J}'_x, \mathbf{J}'_y$ are Jacobian matrices

The impact of effective low rank on the dynamics

Many dynamics on networks have the form

$$\dot{\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$$

with $\boldsymbol{x} \in \mathbb{R}^N$.

Examples:

- SIS (mean-field) : $\dot{x}_i = -d_i x_i + \gamma(1 - x_i)\, y_i$
- Wilson-Cowan: $\dot{x}_i = -d_i x_i + (1 - a x_i)\frac{1}{1+e^{-b(\gamma\, y_i - c)}}$
- Recurrent Neural Networks (RNN): $\dot{x}_i = -d_i x_i + \tanh(\gamma y_i + c_i)$
- Kuramoto-Sakaguchi: $\dot{z}_j = \mathrm{i}\omega_j z_j + \gamma\, e^{-\mathrm{i}\alpha}\, y_j - \gamma\, e^{\mathrm{i}\alpha}\, z_j^2\, \bar{y}_j$ with $z_j = e^{\mathrm{i}\theta_j}$
- Population dynamics: $\dot{x}_i = -d x_i + \gamma x_i y_i$ (Lotka-Volterra)
  $\dot{x}_i = -d x_i - s x_i^2 + \gamma \frac{x_i y_i}{\alpha + y_i}$
  $\dot{x}_i = a - d\, x_i + b x_i^2 - c x_i^3 + \gamma\, x_i\, y_i$

for $i, j \in \{1, ..., N\}$ and $y_i = \sum_{j=1}^{N} W_{ij} x_j$.

Original dynamics

$$\dot{\boldsymbol{x}} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x})$$

Reduced dynamics (with $\boldsymbol{X} = \mathbf{M}\boldsymbol{x}$)

$$\dot{\boldsymbol{X}} = \mathbf{M}\mathbf{g}(\mathbf{M}^{+}\mathbf{X}, \mathbf{W}\mathbf{M}^{+}\mathbf{X})$$

where n is the dimension of the reduced system
$\mathbf{M}^{+}$ denotes the pseudoinverse of $\mathbf{M}$
$\mathbf{M} = \mathbf{V}_n^{\top}$ is n-truncated right singular vector matrix

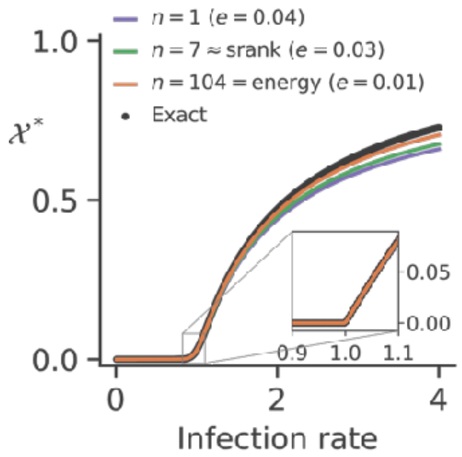
The alignment error is

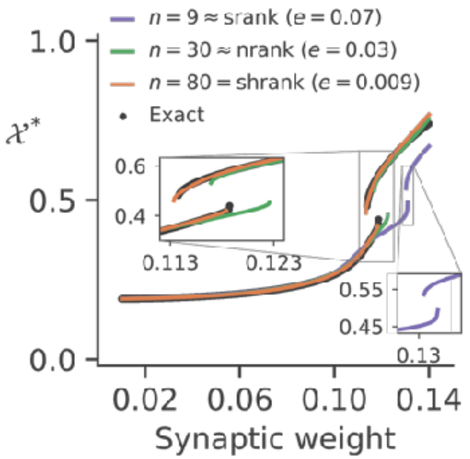
$$\mathcal{E}(x) \leq \frac{1}{\sqrt{n}} \left[\|\mathbf{V}_n^{\top} \mathbf{J}'_x (\mathbf{I} - \mathbf{V}_n \mathbf{V}_n^{\top}) x\| + \sigma_{n+1} \|\mathbf{V}_n^{\top} \mathbf{J}'_y\|_2 \|x\| \right]$$

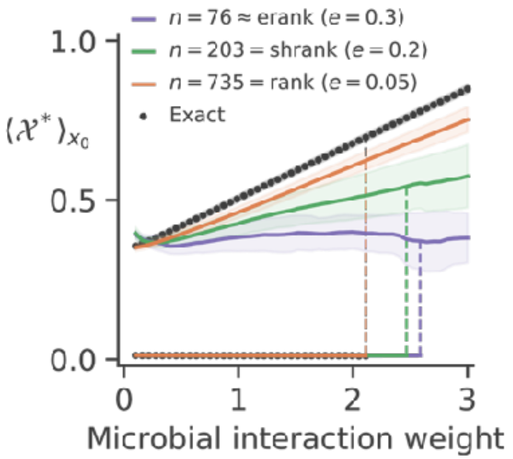
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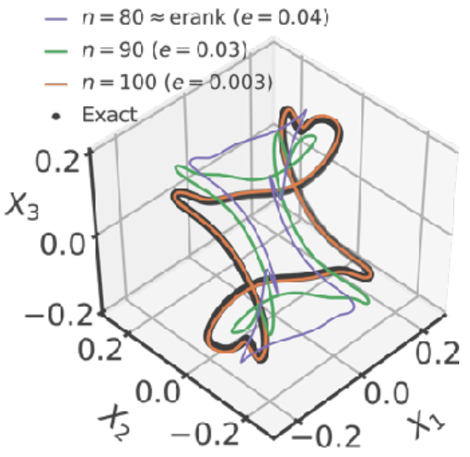
Rapid singular value decrease can induce
rapid alignment error decrease!

Reproduction of the dynamics with increasing accuracy

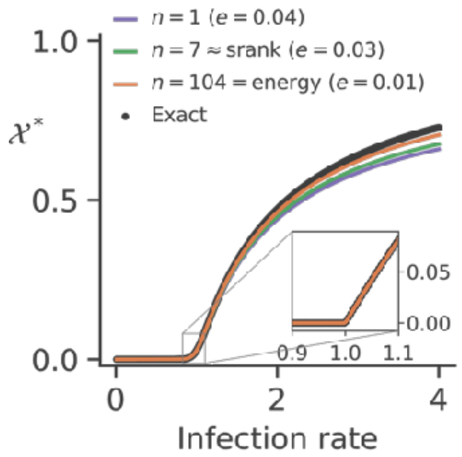


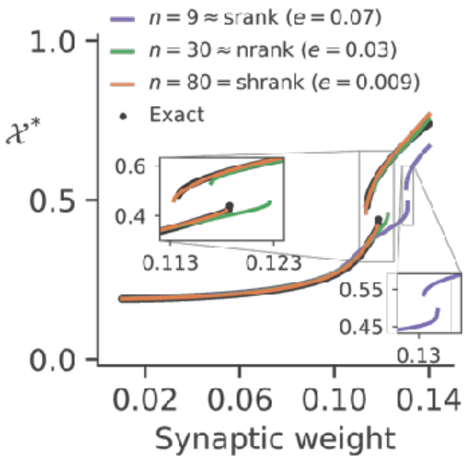


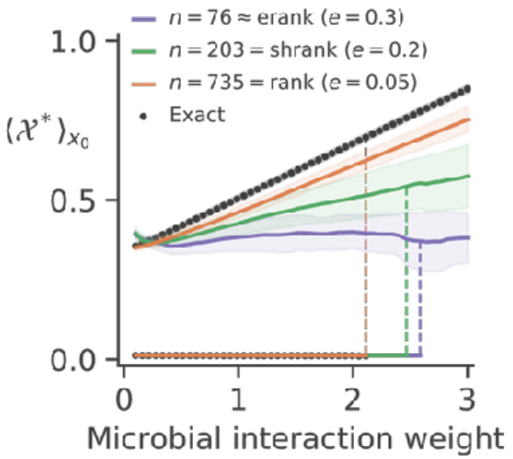


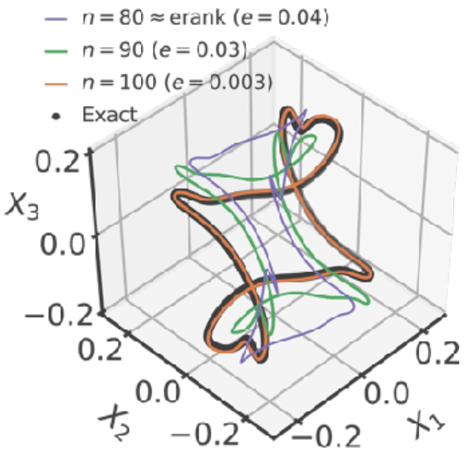


Reproduction of the dynamics with increasing accuracy









Emergence of higher-order interactions

Original Quenched mean-field SIS dynamics

$$\dot{x}_i = -\alpha x_i + \beta(1-x_i)\sum_{j=1}^{N} W_{ij}x_j, \quad i \in \{1,...,N\}$$

Reduced dynamics

$$\dot{X}_\mu = -\alpha X_\mu + \beta\sum_{\nu=1}^{n}\mathcal{W}_{\mu\nu}X_\nu - \beta\sum_{\nu,\tau=1}^{n}\mathcal{W}^{(3)}_{\mu\nu\tau}X_\nu X_\tau, \quad \mu \in \{1,...,n\}$$

Original Kuramoto-Sakaguchi dynamics

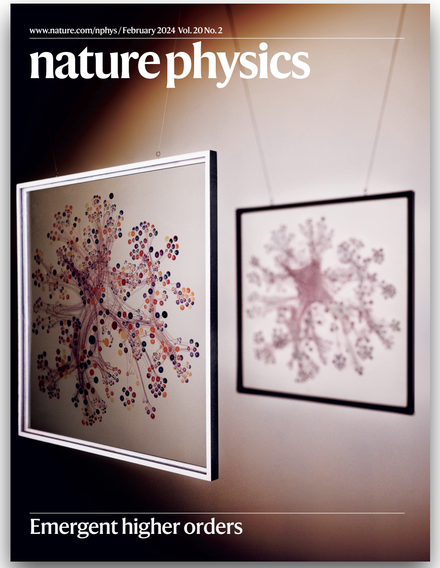
$$\dot{z}_j = i\omega_j z_j + \sum_{k=1}^{N} W_{jk}[z_k e^{-i\alpha} - z_j^2 \bar{z}_k e^{i\alpha}], \quad j \in \{1,...,N\}$$

Reduced dynamics

$$\dot{Z}_\mu = i\sum_{\nu=1}^{n}\Omega_{\mu\nu}Z_\nu + \sum_{\nu=1}^{n}\mathcal{W}^{(2)}_{\mu\nu}Z_\nu e^{-i\alpha} - \sum_{\alpha,\beta,\gamma=1}^{n}\mathcal{W}^{(4)}_{\mu\alpha\beta\gamma}Z_\alpha Z_\beta \bar{Z}_\gamma e^{i\alpha}$$

Main takeaways

- The rapid decrease of the singular values of adjacency matrices (i.e. low effective rank) offers a *justification* for low-dimensional mathematical models beyond mathematical and/or conceptual convenience.
- A large proportion of real networks can be considered as having a low effective rank.
- The higher-order interactions observed in some systems could be a byproduct of a low-dimensional representation used to analyze them.




nature physics

Article

https://doi.org/10.1038/s41567-023-02303-0

The low-rank hypothesis of complex systems

Received: 18 October 2022

Accepted: 24 October 2023

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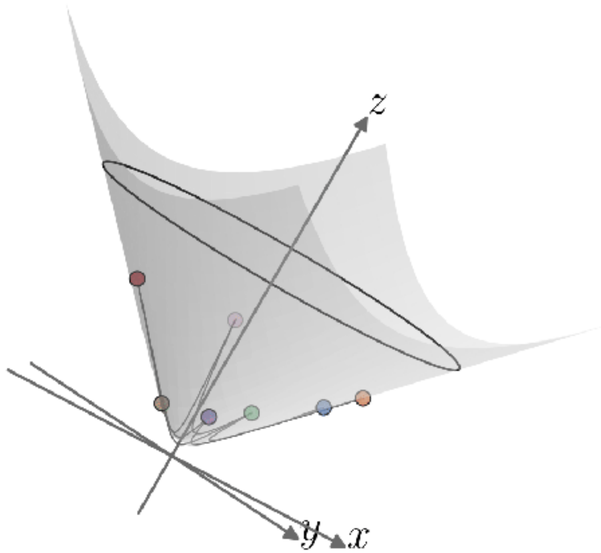
Outline

1. Are simple models enough to study complex systems/networks?

2. **“Simple” ways to encode structural complexity**

   (a) **latent metric space**

   (b) stub types



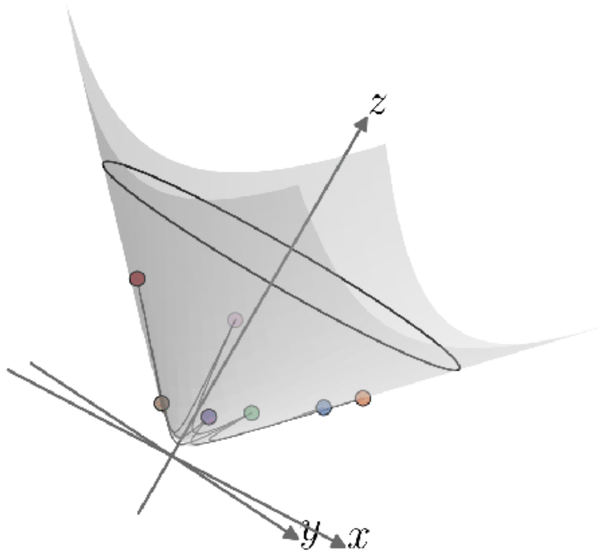
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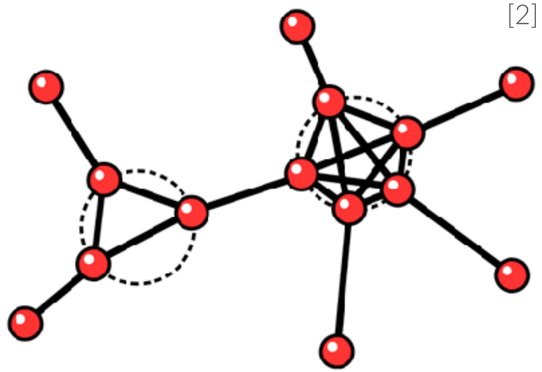


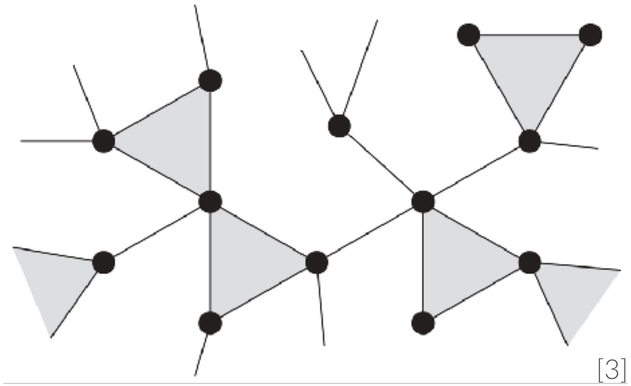
Modeling clustering

Tricky because clustering consists in three-node interactions while our mathematical tools rely on pairwise interactions either explicitly or implicitly.

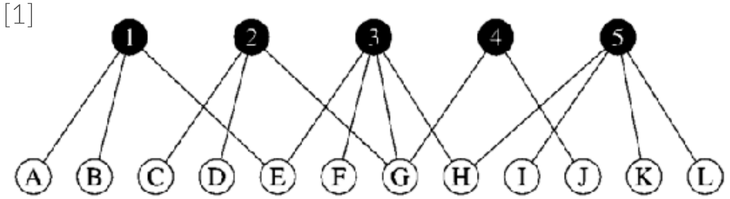
Most models therefore assume

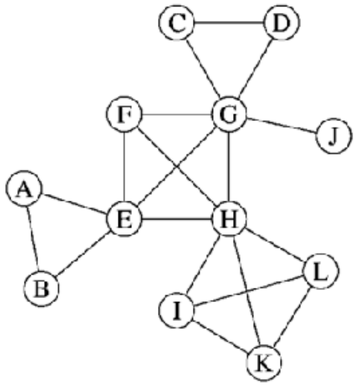
- an underlying tree-like structure
- that the networks are dense

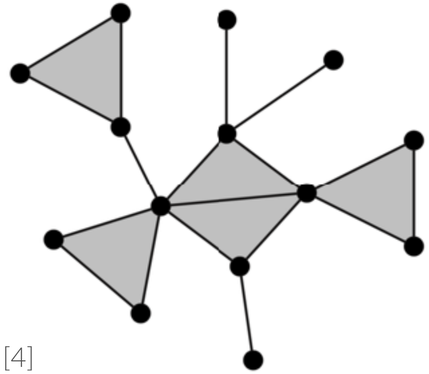

[2]

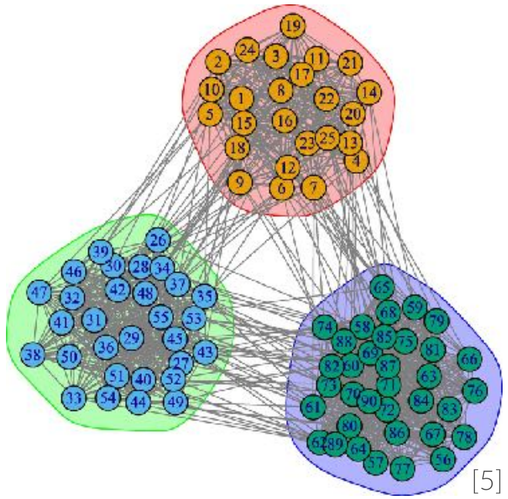

[3]

[1]






[4]


[5]

[1] Phys. Rev. E 68, 026121 (2003)
[2] Phys. Rev. E 80, 036107 (2009)
[3] Phys. Rev. Lett. 103, 058701 (2009)
[4] Phys. Rev. E 82, 066118 (2010)
[5] Appl. Netw. Sci. 4, 122 (2019)

A geometric approach to clustering

Assume that the nodes are embedded in a metric space and that any two nodes are connected with a probability that is a decreasing function of the distance between them.

For further info, see Phys. Rep. 499, 1–101 (2011)

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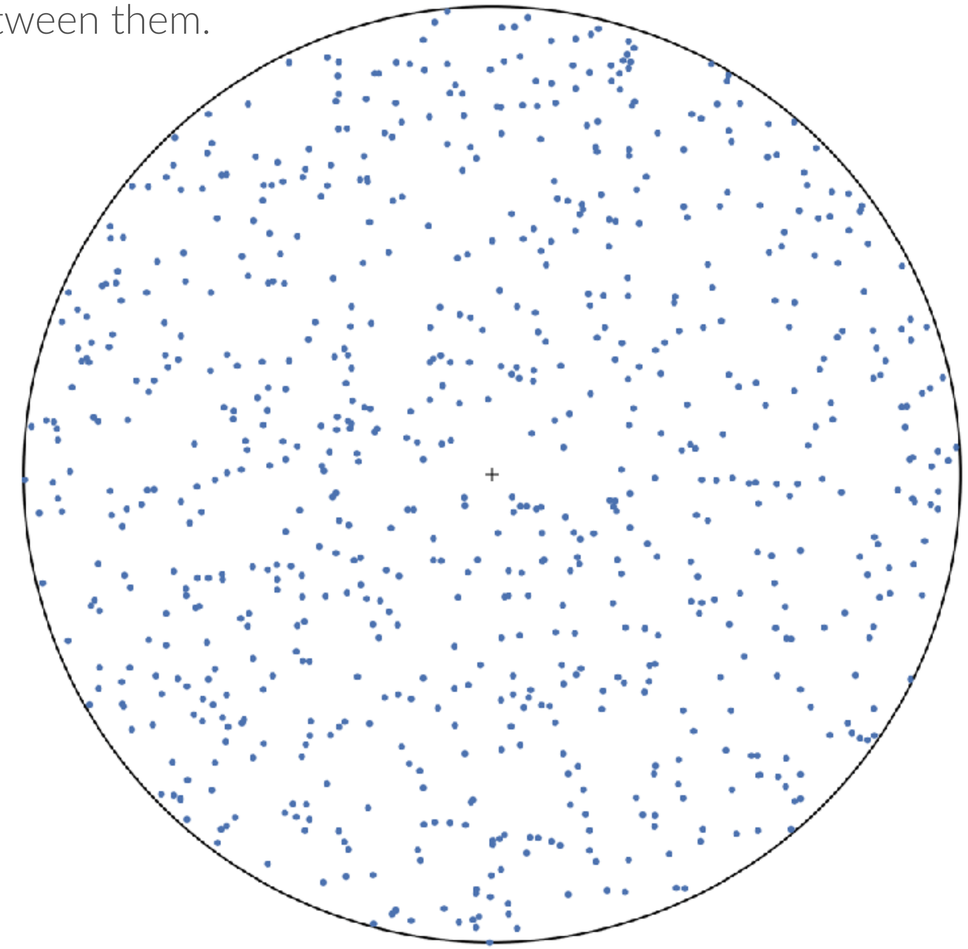
Properties of any metric space

Identity of indiscernibles	$d(x,y) = 0 \quad \Leftrightarrow \quad x = y$
Non-negativity	$d(x,y) \geq 0$
Symmetry	$d(x,y) = d(y,x)$
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Simple random geometric graph

1. Sprinkle N nodes uniformly on a disk of radius R.
2. Connect any nodes separated by a distance less than r.

✓ high clustering

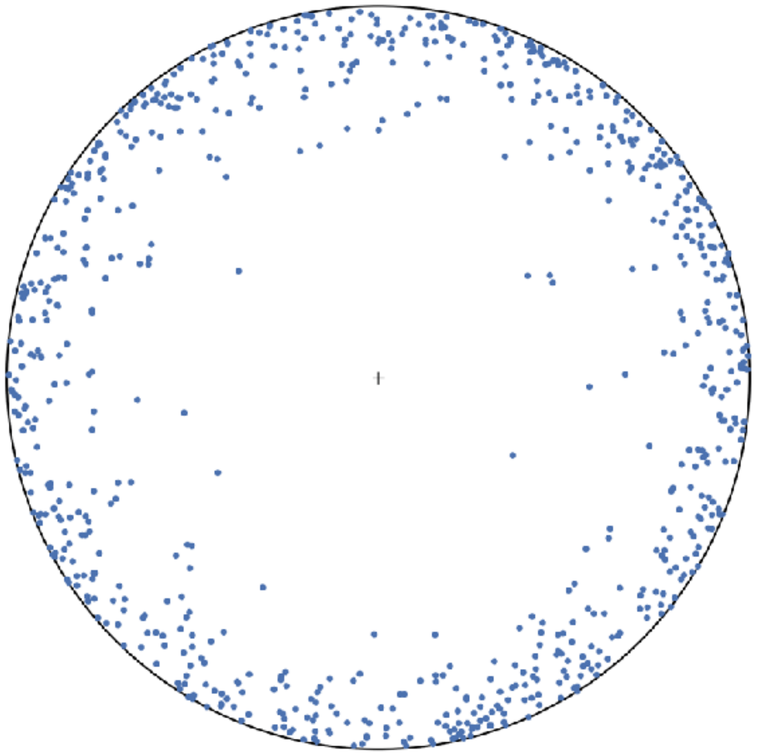
✗ binomial/Poisson degree distribution

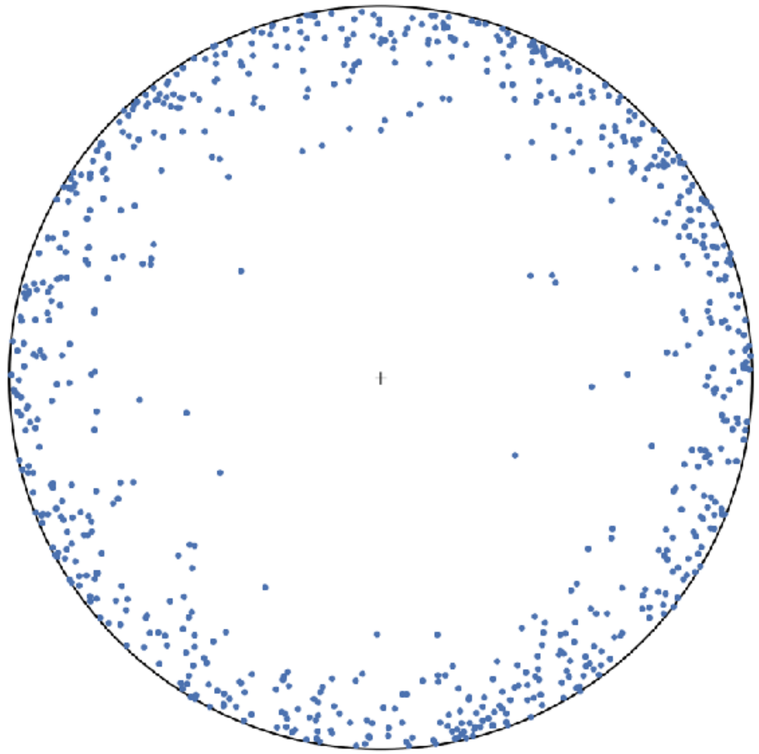
For further info, see Phys. Rep. 499, 1–101 (2011)

A geometric approach to clustering: Hyperbolic geometry

Simple random geometric graph

1. Sprinkle N nodes uniformly on the hyperbolic disk of radius R.
2. Connect any nodes separated by a distance less than $r = R$.





✓ high clustering

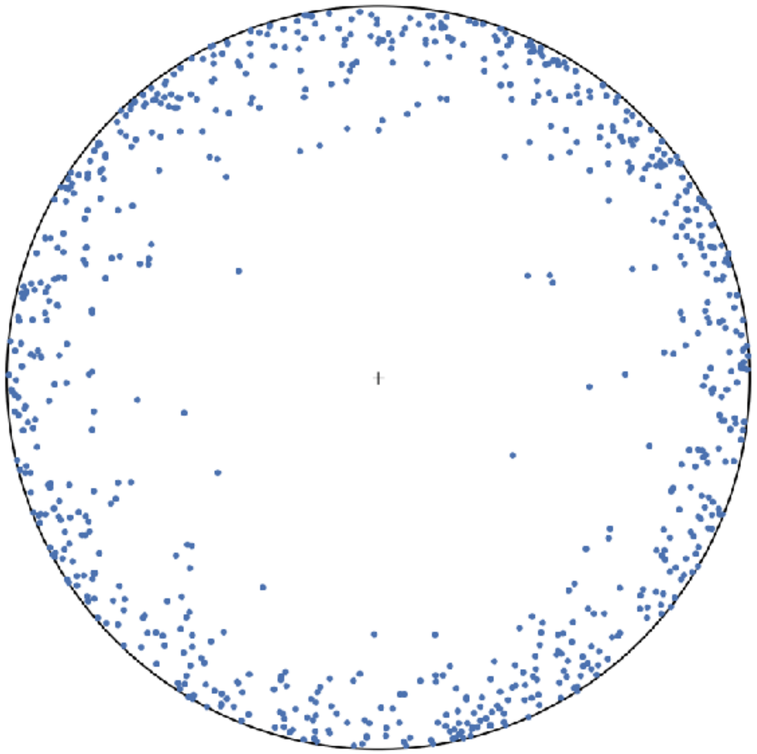
✓ power-law degree distribution with exponent -3

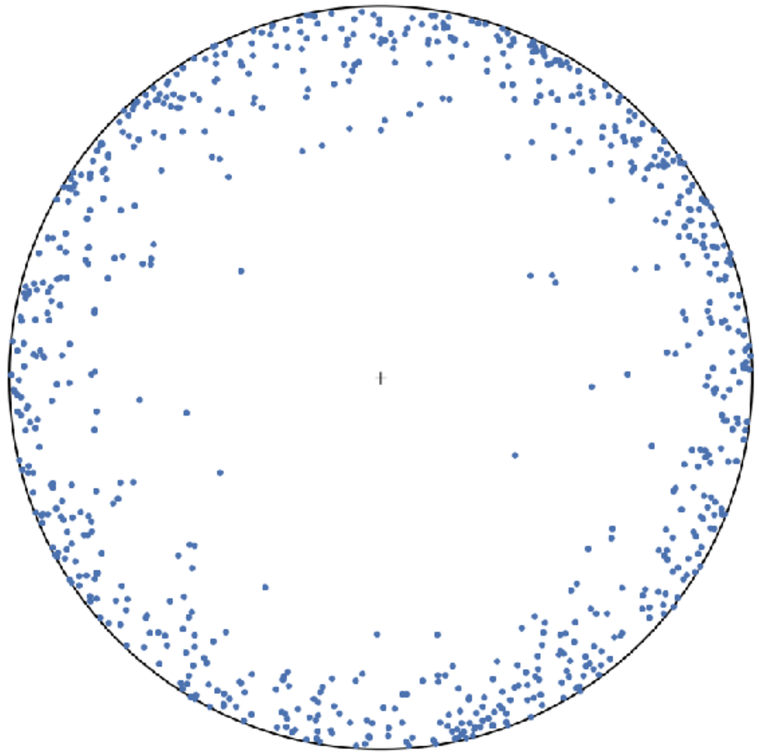
Phys. Rev. E 82, 036106 (2010)

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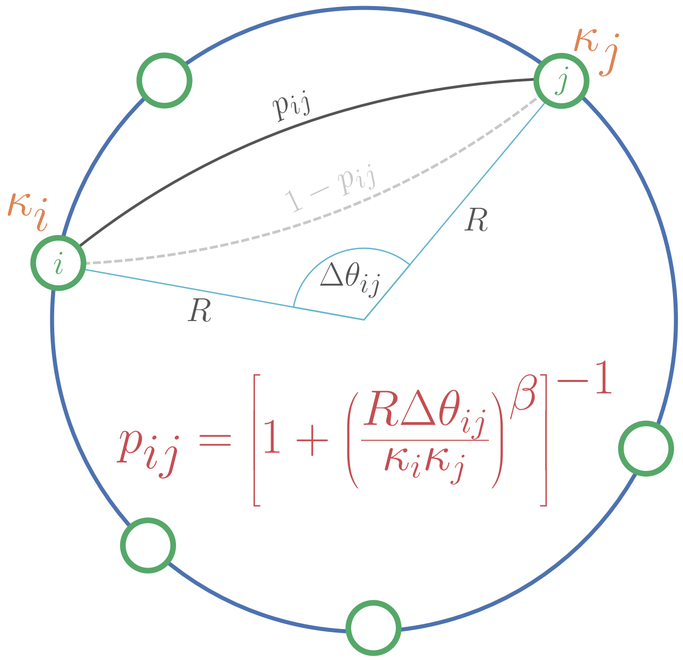


✓ high clustering

✓ power-law degree distribution with exponent -3

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A geometric approach to clustering: the $\mathbb{S}^1/\mathbb{H}^2$ model



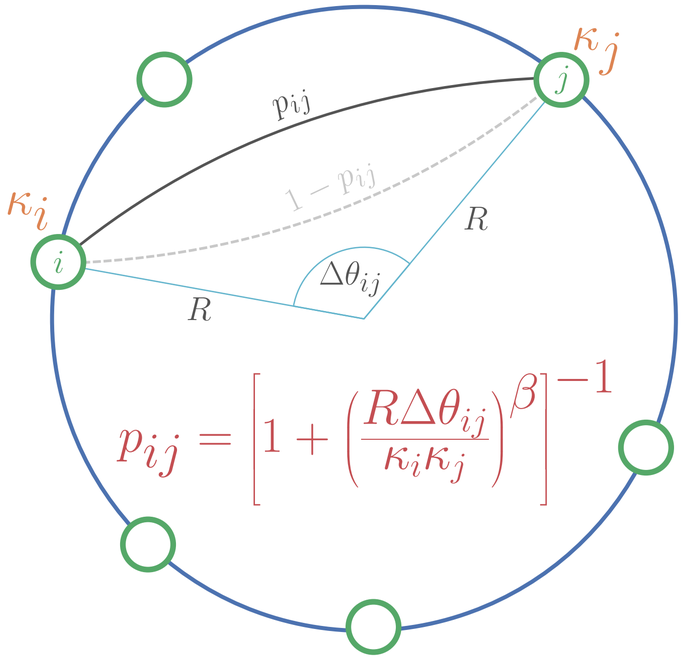
The $\mathbb{S}^1$ model

1. Sprinkle N nodes uniformly on a circle of radius R.
2. Assign an expected degree κ to each node according to some pdf $\rho(\kappa)$.
3. Draw a link between node i and node j with probability p_{ij}.

★ fixes the expected degree of nodes (κ) → soft configuration model (CM)
★ triangle inequality of the underlying metric space → triangles from pairwise interactions
★ level of clustering tuned with parameter β

[1] Phys. Rev. E 80, 035101 (2009)
[2] Phys. Rev. E 82, 036106 (2010)
[3] Phys. Rev. Lett. 100, 078701 (2008)
[4] Nat. Rev. Phys. 3, 114 (2021)
[5] Nat. Commun. 8, 14103 (2017)
[6] Phys. Rev. E 84, 026114 (2011)
[7] Phys. Rev. E 95, 032309 (2017)
[8] Mol. Biosyst. 8, 843 (2012)
[9] Nat. Phys. 12, 1076 (2016)
[10] Phys. Rev. Lett. 118, 218301 (2017)
[11] Nature 489, 537 (2012)
[12] Sci. Rep. 5, 9421 (2015)
[13] J. Stat. Phys. 173, 775 (2018)
[14] New J. Phys. 20, 052002 (2018)
[15] New J. Phys. 21, 123033 (2019)
[16] Nat. Commun. 8, 1615 (2017)
[17] Nat. Commun. 1, 62 (2010)
[18] PNAS 117, 20244 (2020)

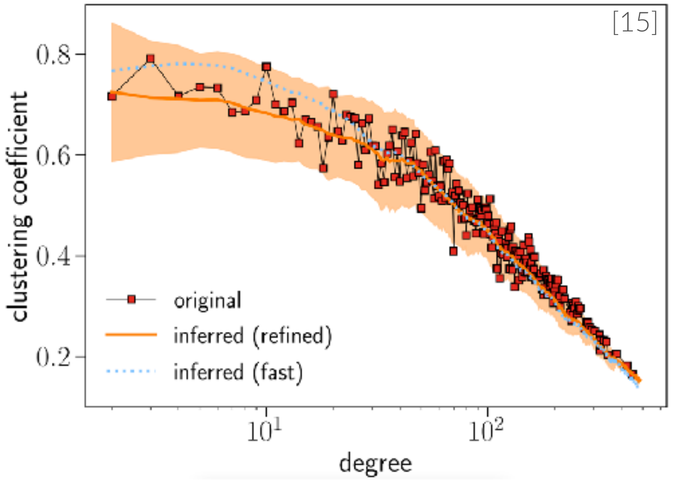
A geometric approach to clustering: the $\mathbb{S}^1/\mathbb{H}^2$ model



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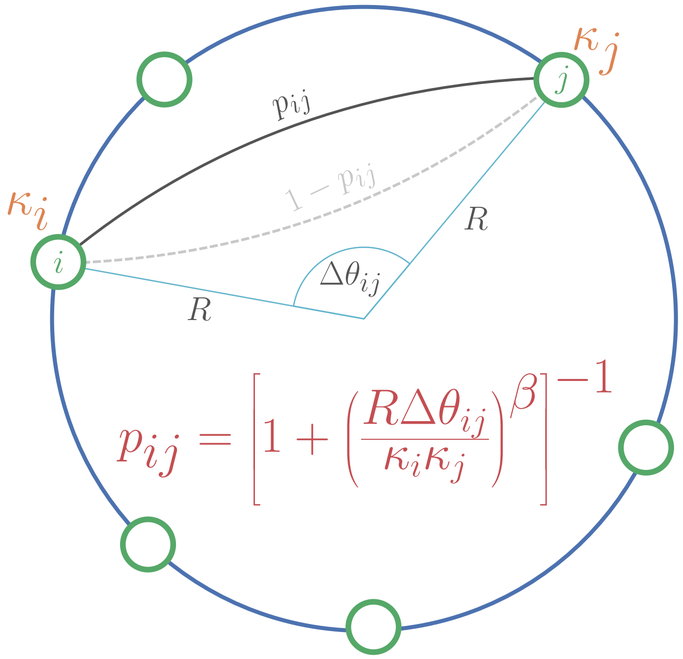
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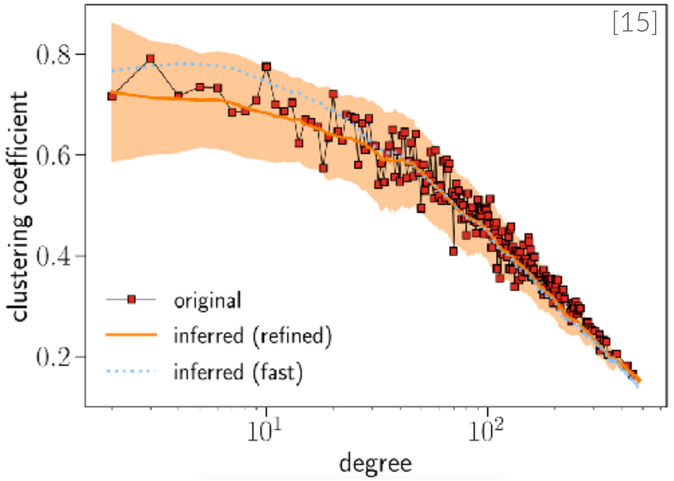
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A geometric approach to clustering: the $\mathbb{S}^1/\mathbb{H}^2$ model





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Other properties and generalizations

- ▷ Amenable to many analytical calculations
- ▷ Geometric interpretation in terms of hyperbolic geometry (the $\mathbb{H}^2$ model) [1,2]
- ▷ Parsimonious explanation of self-similarity [3,4]
- ▷ Generalizable to weighted [5], bipartite [6,7,8], multiplex [9,10] and growing [11] networks
- ▷ Generalizable to networks with community structure [12,13,14]
- ▷ Mapping of real complex networks unto hyperbolic space [15,16]
- ▷ Identification of biochemical pathways in E. Coli [8]
- ▷ Efficient Internet routing protocols [17]
- ▷ Multiscale organization of the human connectome [18]
- ▷ Geometrical interpretation of preferential attachment [11]
- ▷ ...

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Three challenges in modeling directed networks

Properties of any metric space

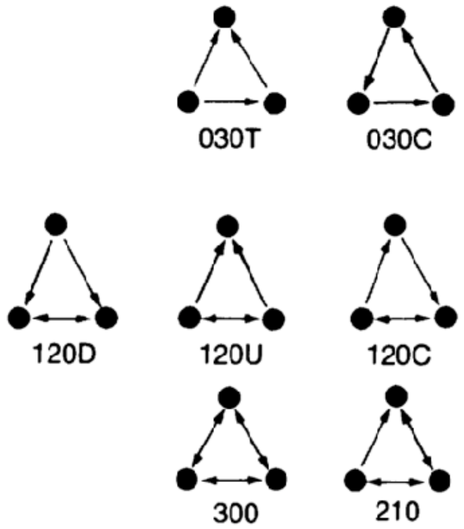
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Clustering : 7 cycles of length 3



Adapted from Holland & Leinhardt. Local Structure in Social Networks. Sociol. Methodol., 7, 1–45 (1976)

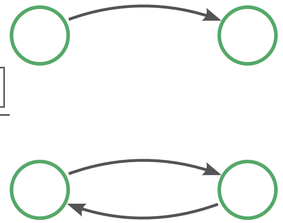
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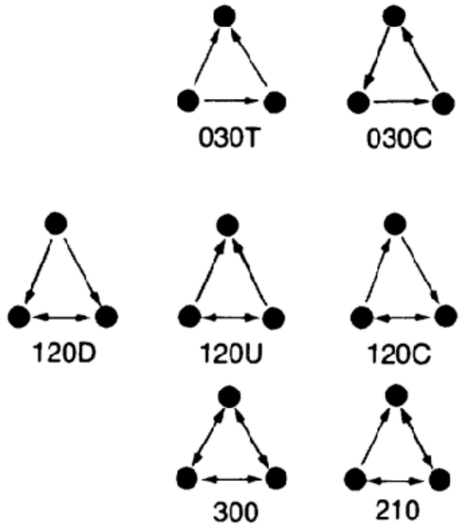
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Triangle inequality	$d(x,y) \leq d(x,z) + d(z,y)$

Reciprocity : cycles of length 2

$$r = \frac{L^{\leftrightarrow}}{L} = \frac{[\text{number of reciprocal links}]}{[\text{number of links}]}$$

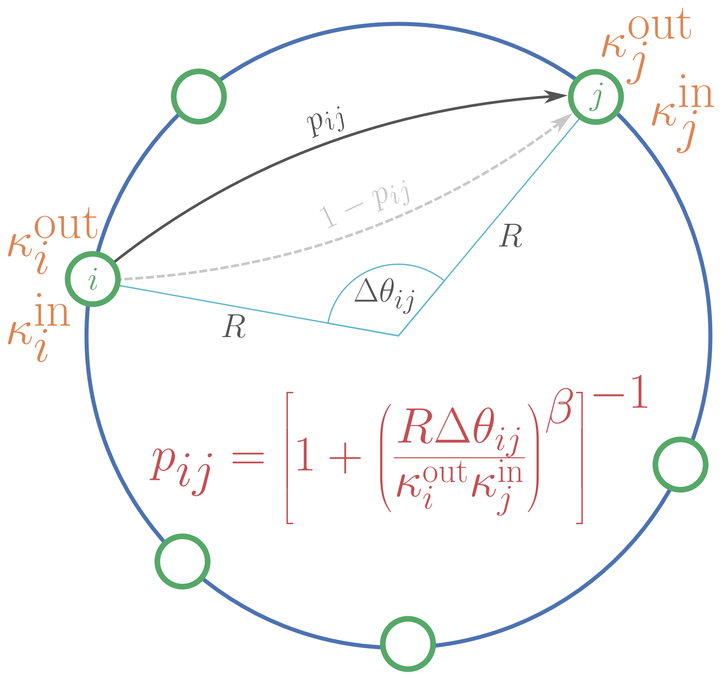


Clustering : 7 cycles of length 3



Adapted from Holland & Leinhardt. Local Structure in Social Networks. Sociol. Methodol., 7, 1–45 (1976)

The directed $\mathbb{S}^1$ model: A straightforward generalization



The directed $\mathbb{S}^1$ model

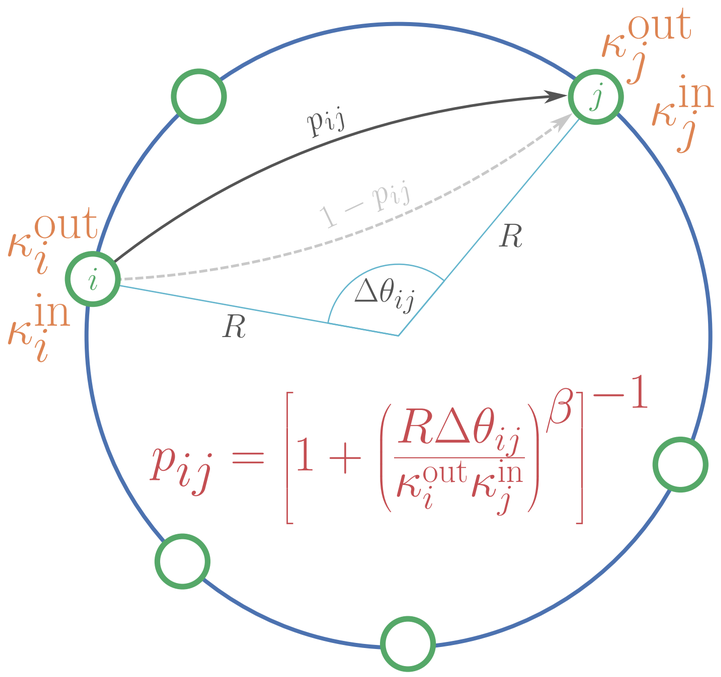
1. Sprinkle N nodes uniformly on a circle of radius R.
2. Assign an expected in-degree κ^{in} and out-degree κ^{out} to each node according to some pdf $\rho(\kappa^{\text{in}}, \kappa^{\text{out}})$.
3. Draw a link from node i to node j with probability p_{ij}.

★ fixes the expected in-degree and out-degree of nodes (κ^{in}, κ^{out}) → soft directed CM

★ triangle inequality of the underlying metric space → triangles from pairwise interactions

★ level of clustering tuned with parameter β

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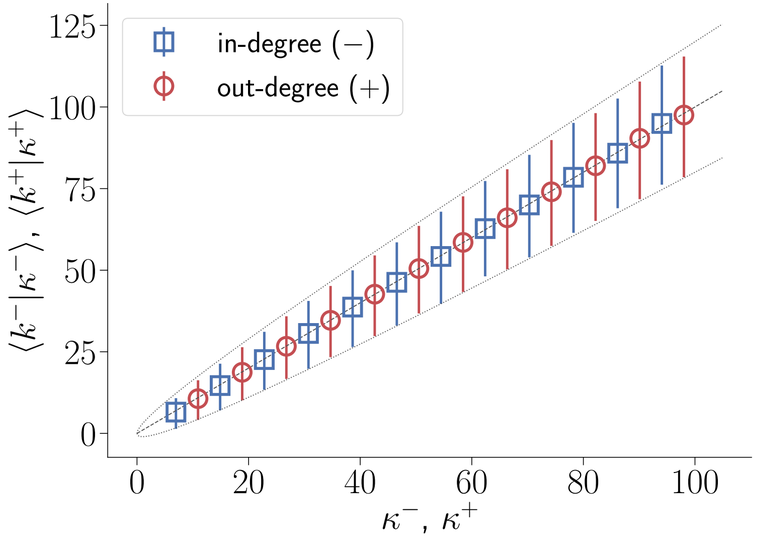
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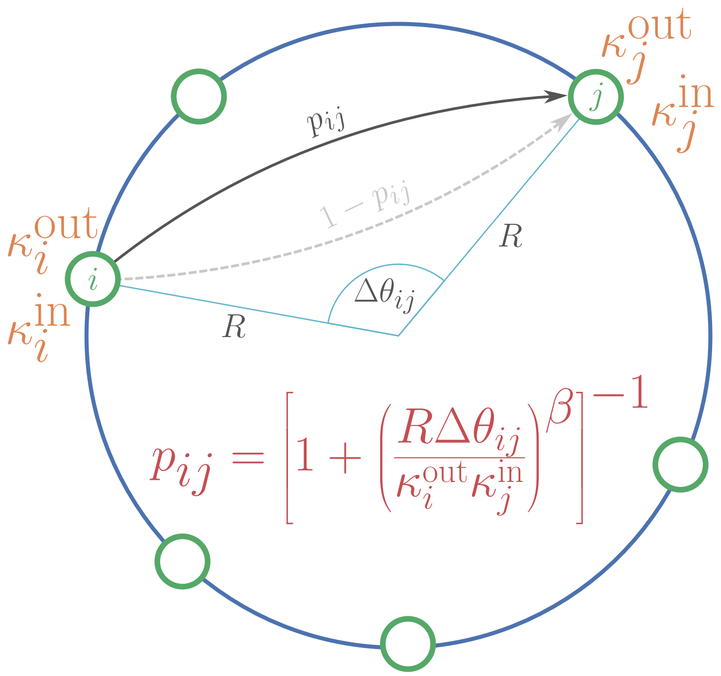
$$\mathbb{E}\left[k^{\text{in}}\middle|\kappa^{\text{in}}\right] \simeq \kappa^{\text{in}}$$

$$\mathbb{E}\left[k^{\text{out}}\middle|\kappa^{\text{out}}\right] \simeq \kappa^{\text{out}}$$

$$P(k^{\text{in}}, k^{\text{out}}) \simeq \iint \frac{[\kappa^{\text{in}}]^{k^{\text{in}}} \mathrm{e}^{-\kappa^{\text{in}}}}{k^{\text{in}}!} \frac{[\kappa^{\text{out}}]^{k^{\text{out}}} \mathrm{e}^{-\kappa^{\text{out}}}}{k^{\text{out}}!} \times \rho(\kappa^{\text{in}}, \kappa^{\text{out}}) d\kappa^{\text{in}} d\kappa^{\text{out}}$$



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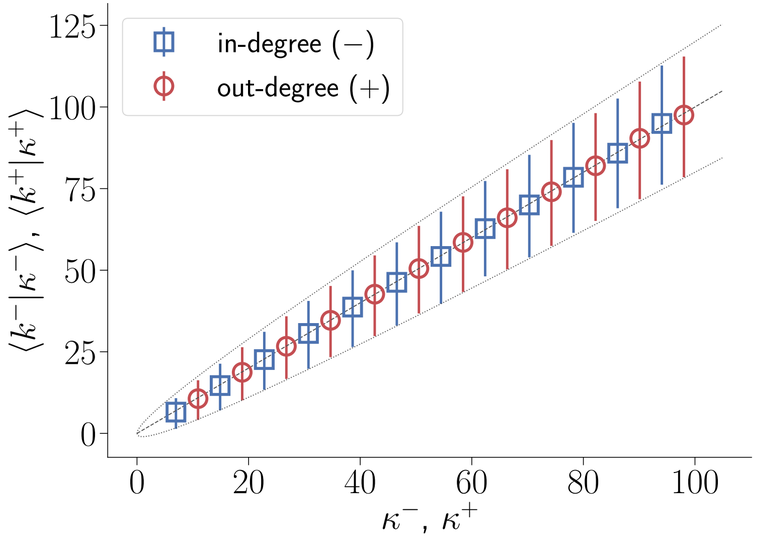
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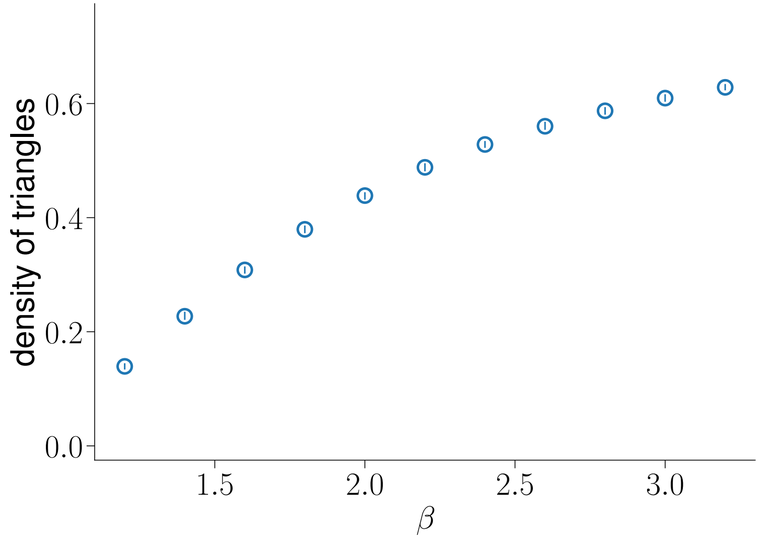
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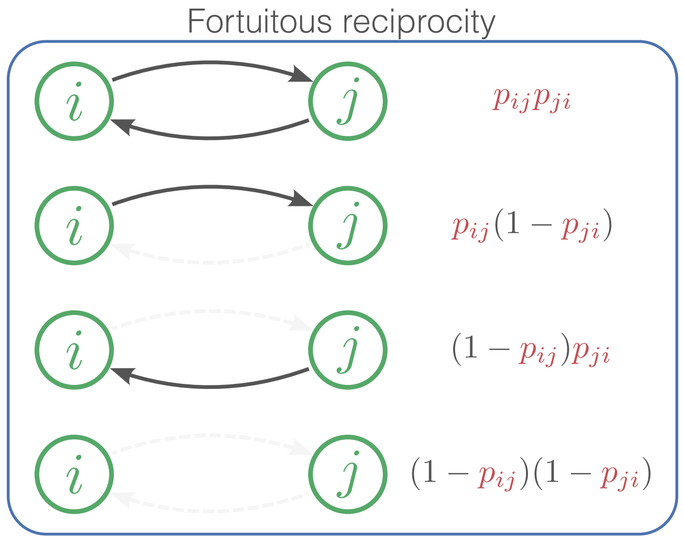
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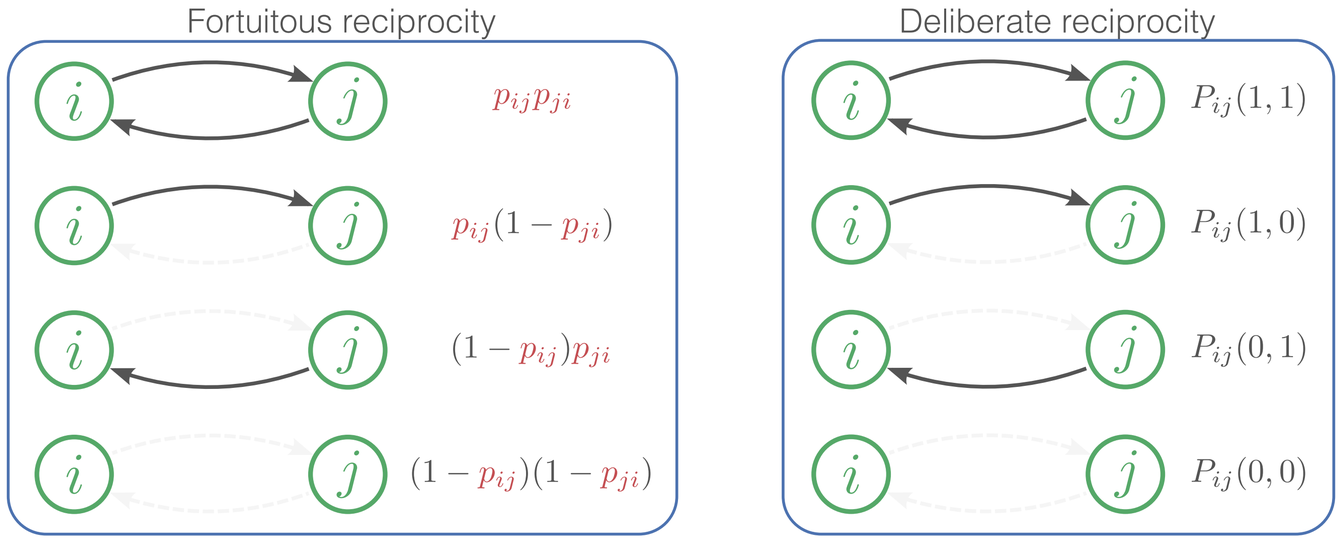
The directed-reciprocal $\mathbb{S}^1$ model: A new link between connections and distances

A random network model defines the probability p_{ij} for a directed link to exist from node i to node j.



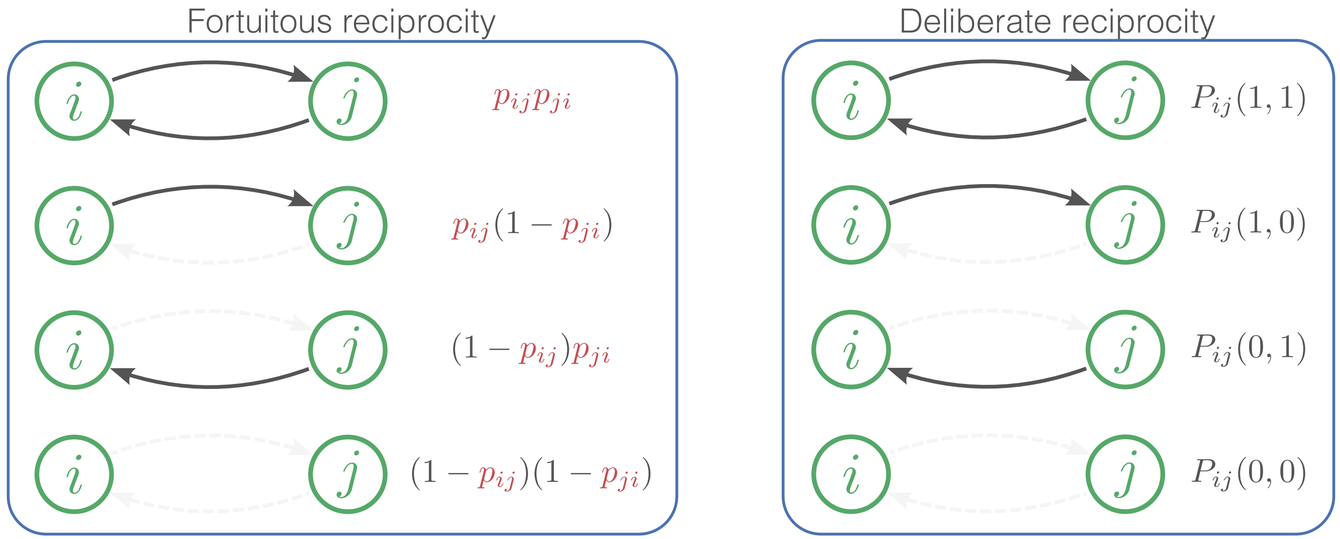
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Level of reciprocity controlled with parameter $-1 \leq \nu \leq 1$

$$P_{ij}(1,1) = \begin{cases} (1-\nu)p_{ij}p_{ji} + \nu \min\{p_{ij}, p_{ji}\} & 0 \leq \nu \leq 1 \\ (1+\nu)p_{ij}p_{ji} + \nu(1-p_{ij}-p_{ji})H(p_{ij}+p_{ji}-1) & -1 \leq \nu \leq 0 \end{cases}$$

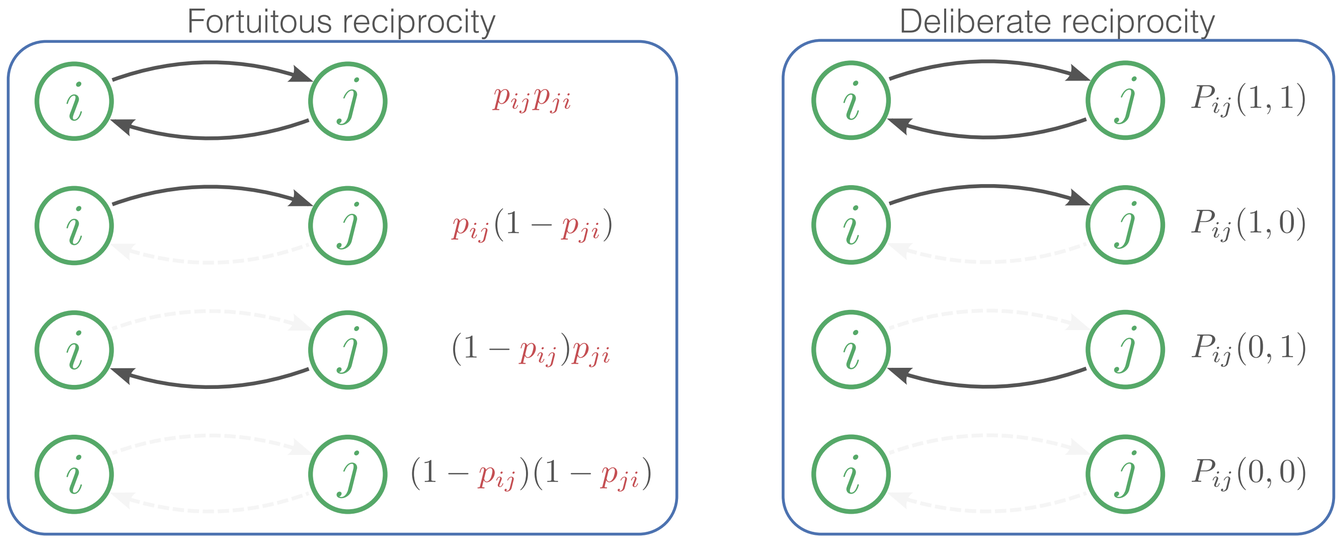
$\nu = 1$: maximal reciprocity

$\nu = 0$: fortuitous reciprocity

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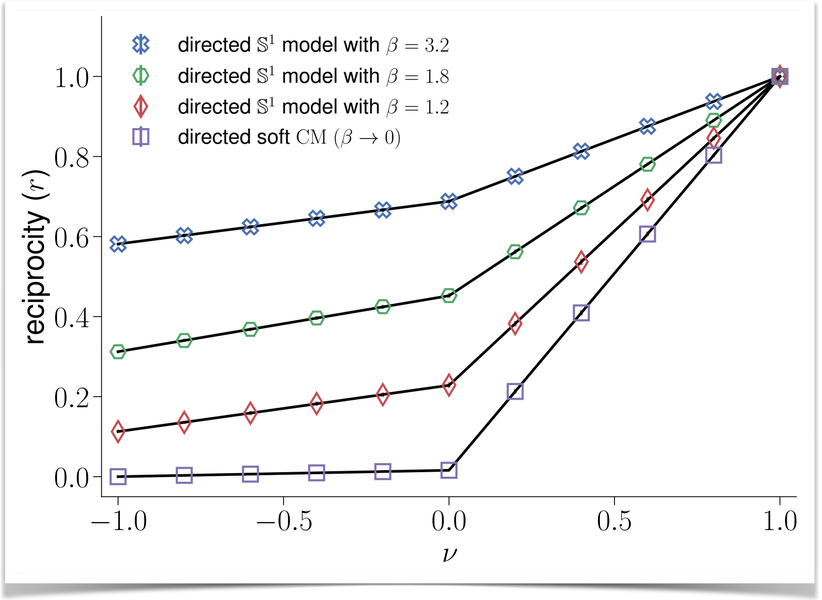
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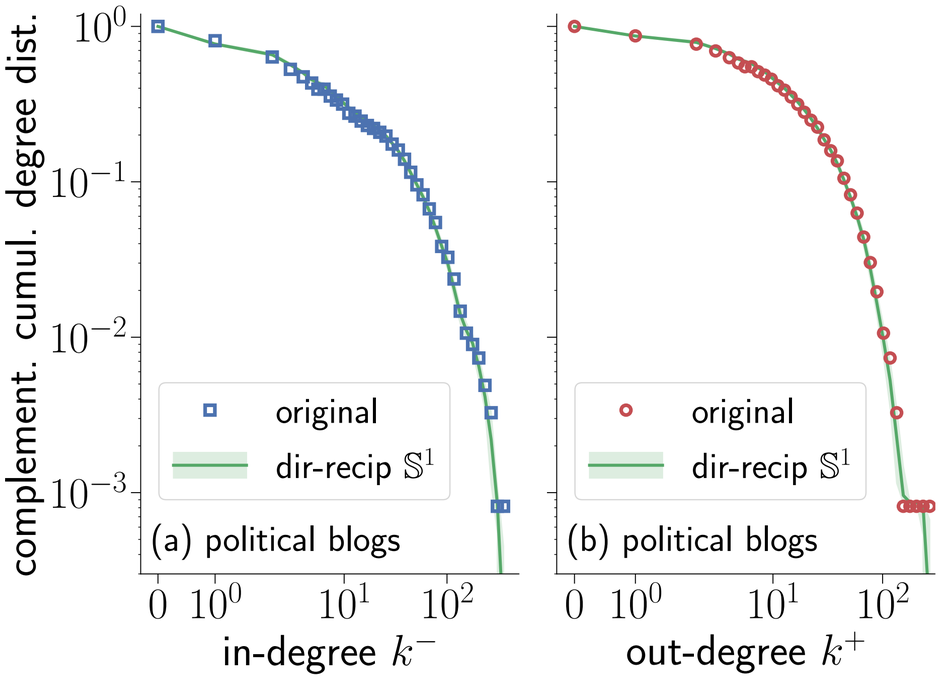
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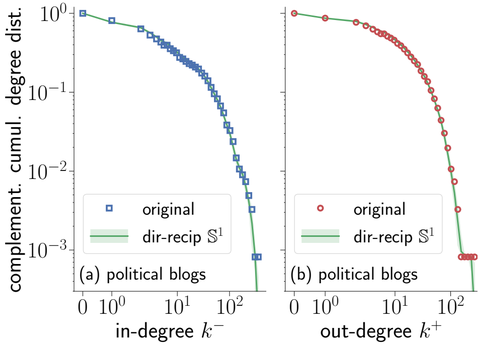
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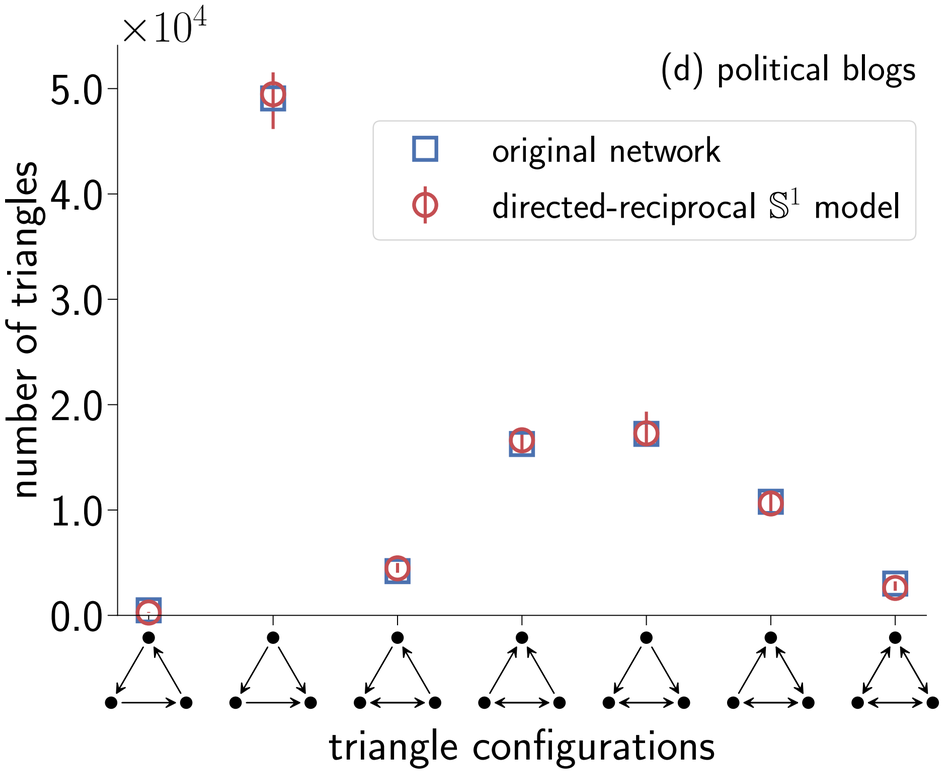


Fitting the directed-reciprocal $\mathbb{S}^1$ model to real networks

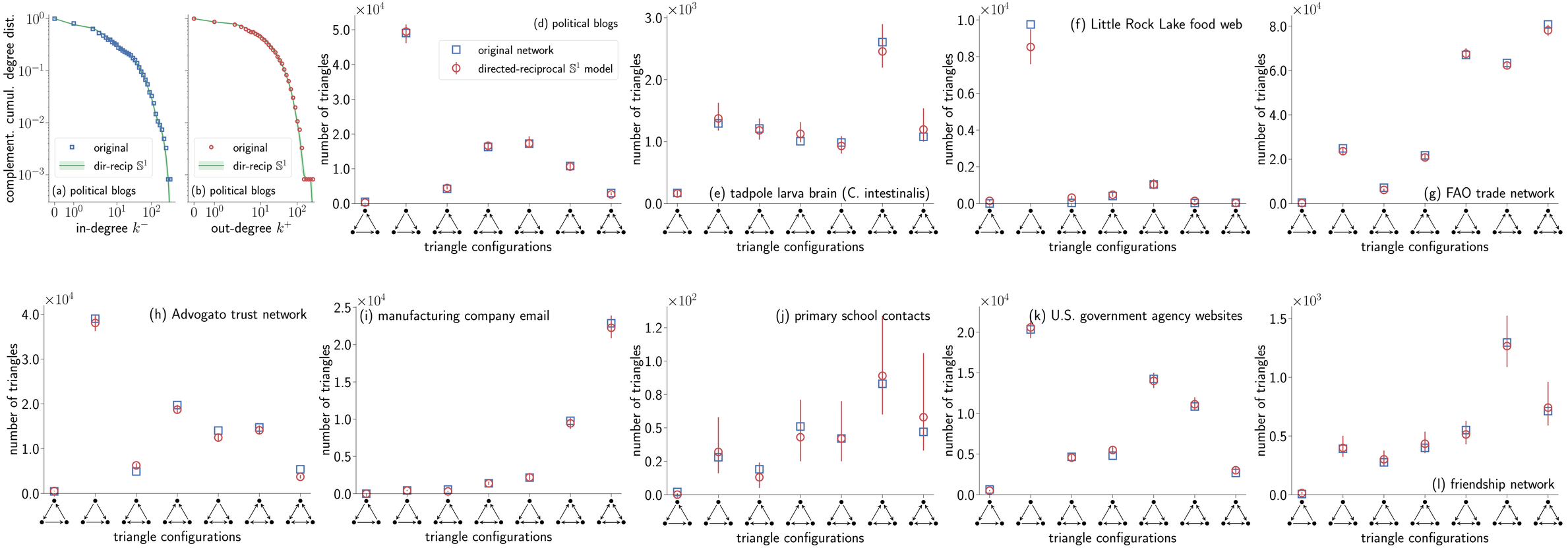


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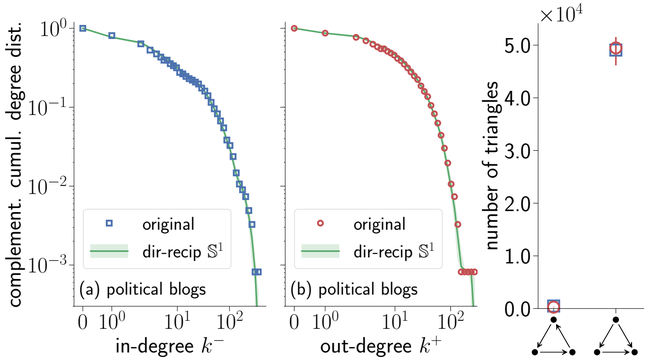


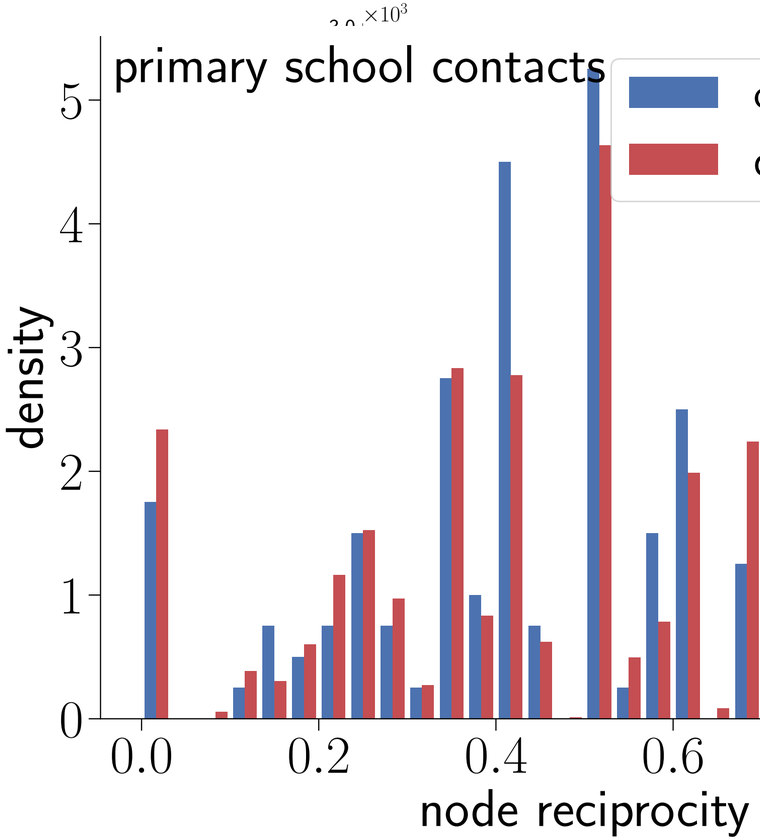


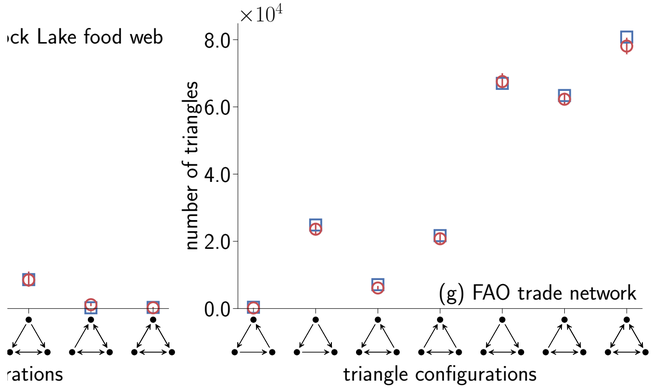
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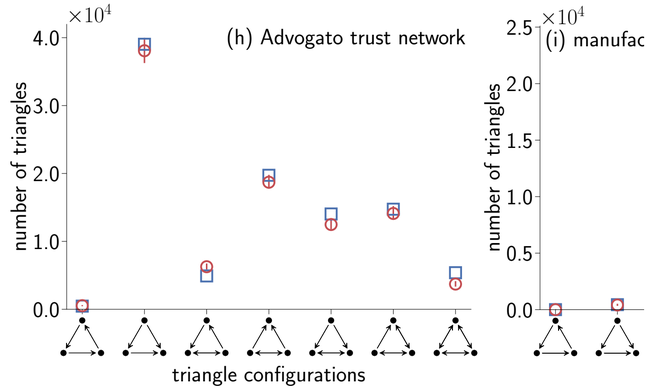


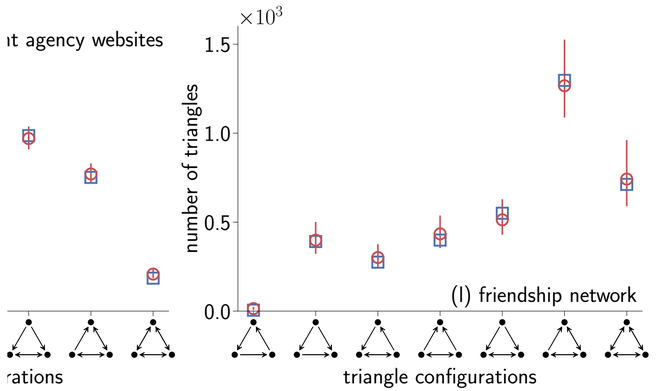
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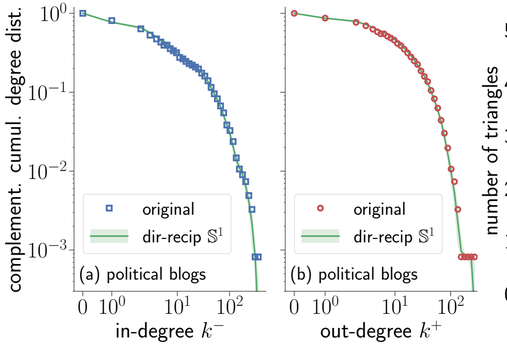


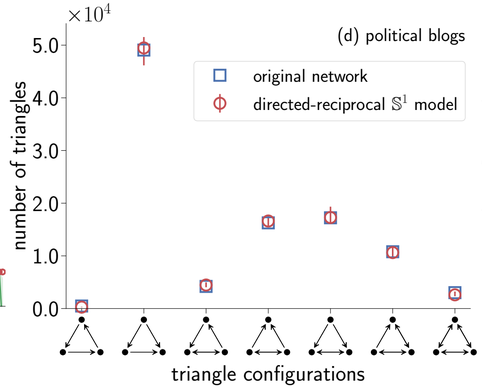


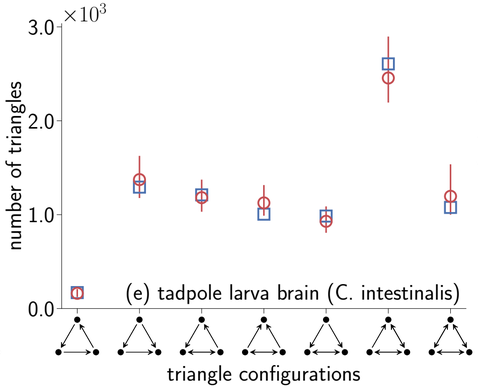


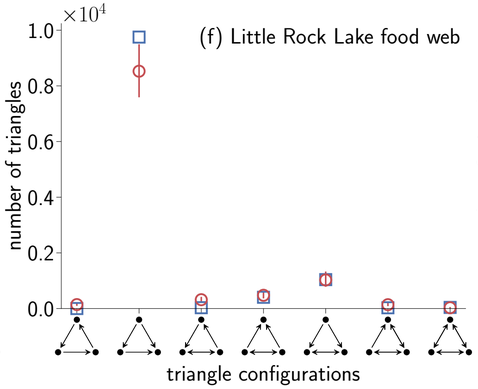


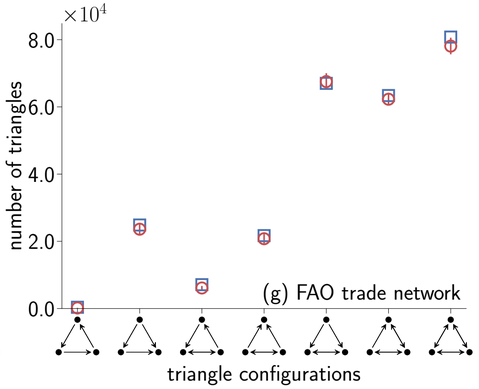
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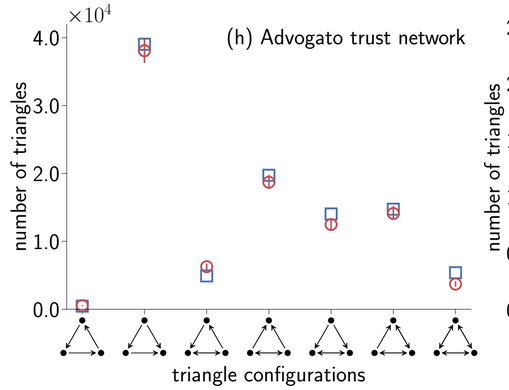


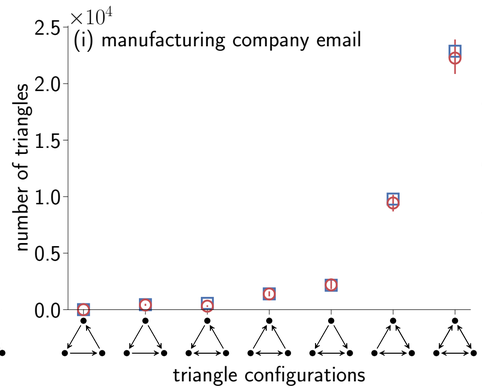


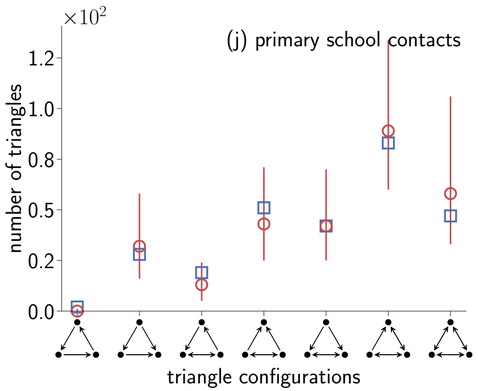


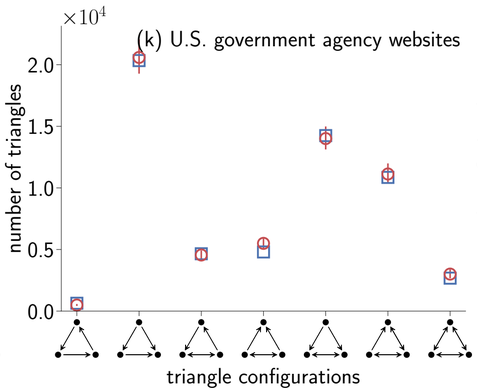


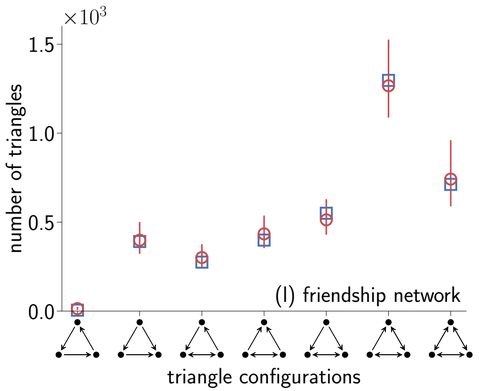


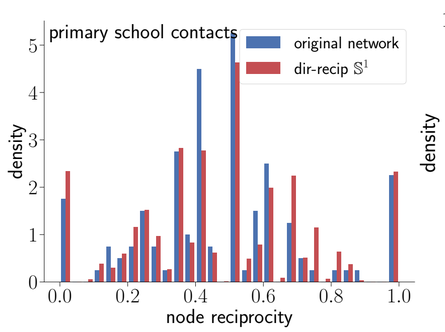


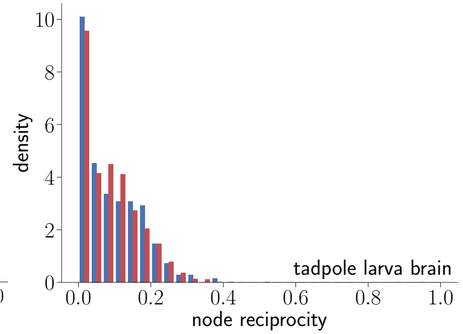




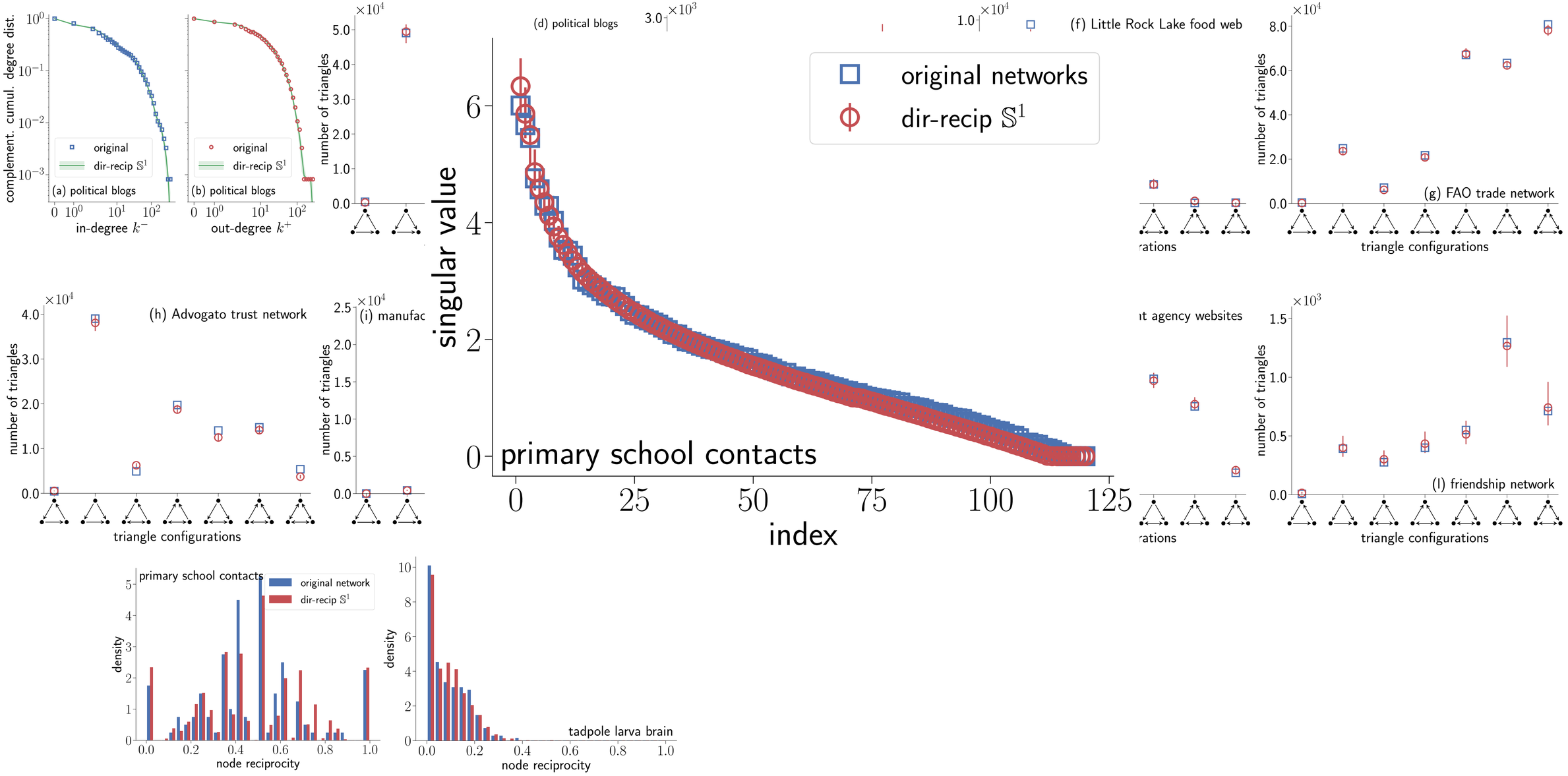




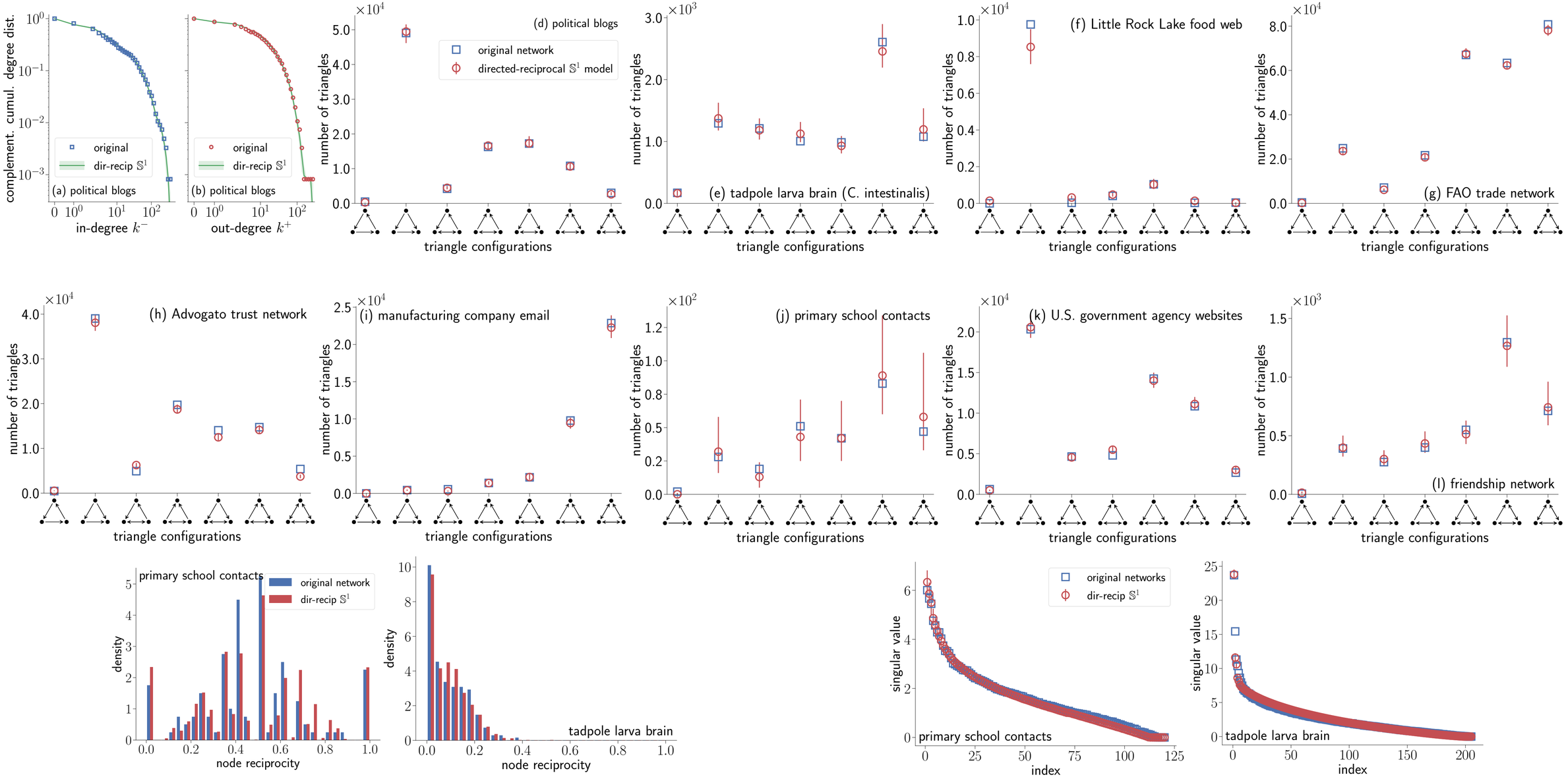




Fitting the directed-reciprocal $\mathbb{S}^1$ model to real networks



Fitting the directed-reciprocal $\mathbb{S}^1$ model to real networks



Main takeaways

- Rethinking the link between connections and distances allows the extension of the network geometry framework to directed networks.
- Assuming an underlying geometry is a simple modeling approach that can reproduce several complex topological properties observed in real networks.


nature physics

Article

https://doi.org/10.1038/s41567-023-02246-6

Geometric description of clustering in directed networks

Received: 27 March 2023

Accepted: 13 September 2023

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Hyperbolic Embedding of Directed Networks

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Preprint on arXiv expected this fall.

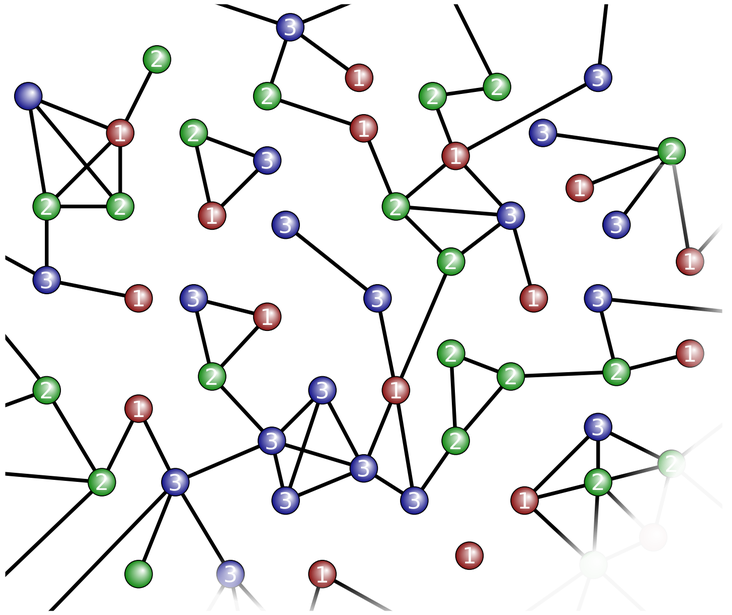
Outline

1. Are simple models enough to study complex systems/networks?

2. **“Simple” ways to encode structural complexity**

   (a) latent metric space

   (b) stub types



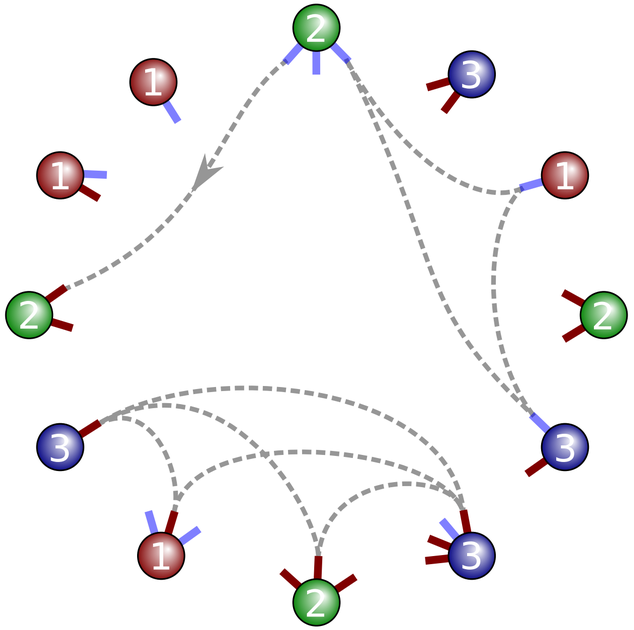
General *stub matching* scheme

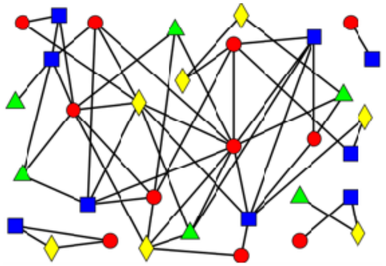
Many network models can be seen as a *stub matching* scheme with

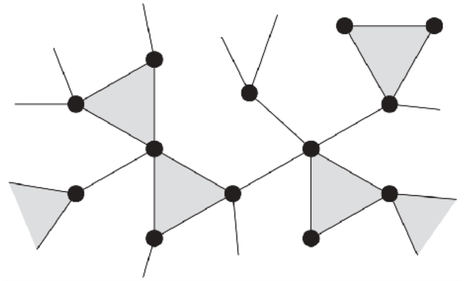
- node types
- stub types
- rules governing how stubs are matched

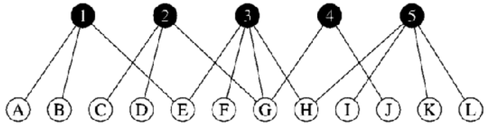
This perspective facilitates the mathematical description of the dynamical processes on networks

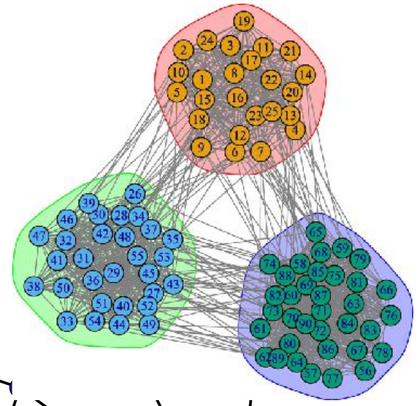
- probability generating functions (ex.: percolation, robustness)
- ordinary differential equations (ex.: epidemic spreading, opinion dynamics)

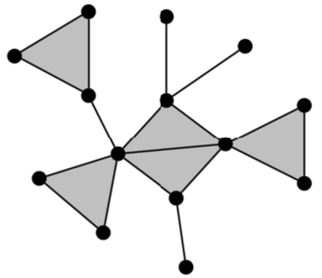


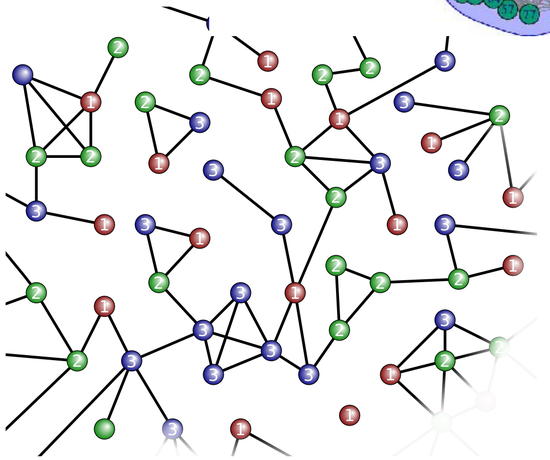












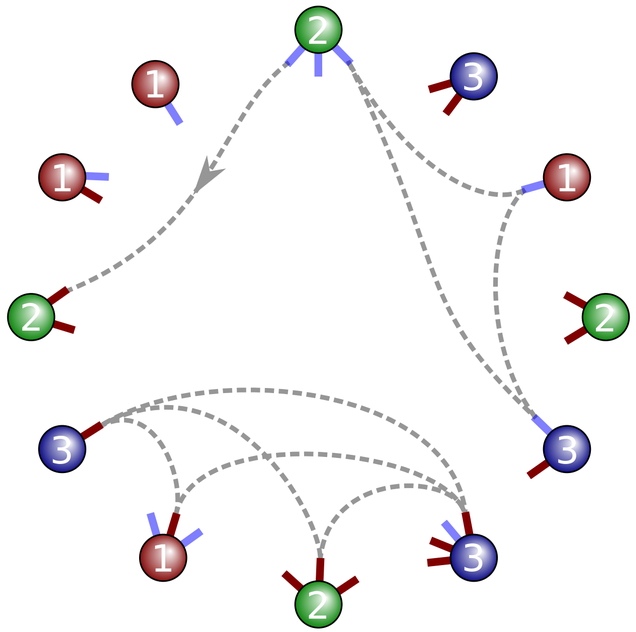
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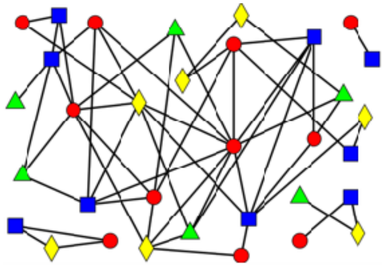
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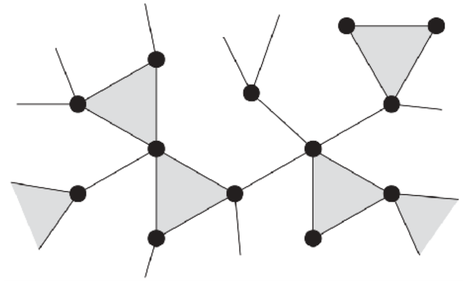
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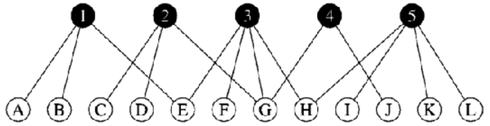
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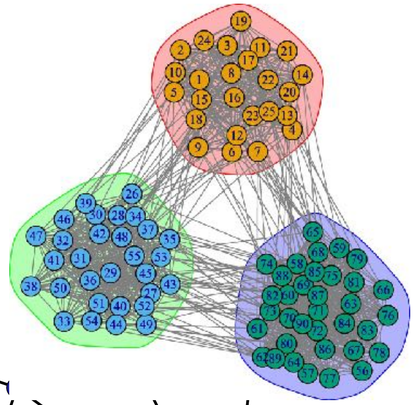
- probability generating functions (ex.: percolation, robustness)
- ordinary differential equations (ex.: epidemic spreading, opinion dynamics)

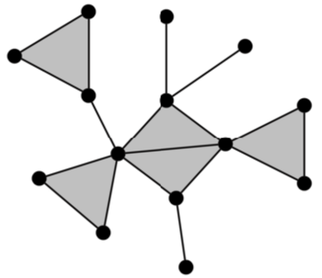


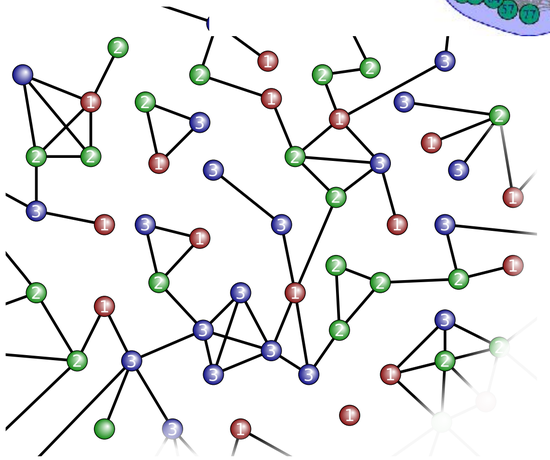










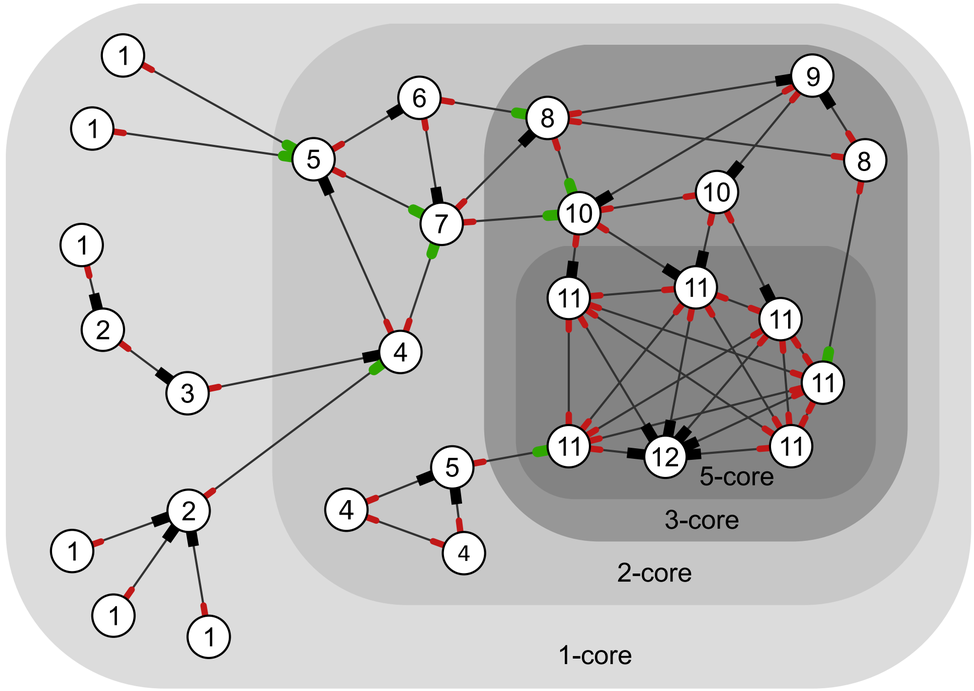


Most of these approaches use stub types to enforce *local* connection patterns.

Mesoscopic level: The k-core/onion decomposition

Onion decomposition: k-core decomposition with additional information about the positions of nodes within every k-shell (layers).

Information about layers is obtained from the k-core decomposition with minimal additional computational cost.



Mesoscopic level: The k-core/onion decomposition

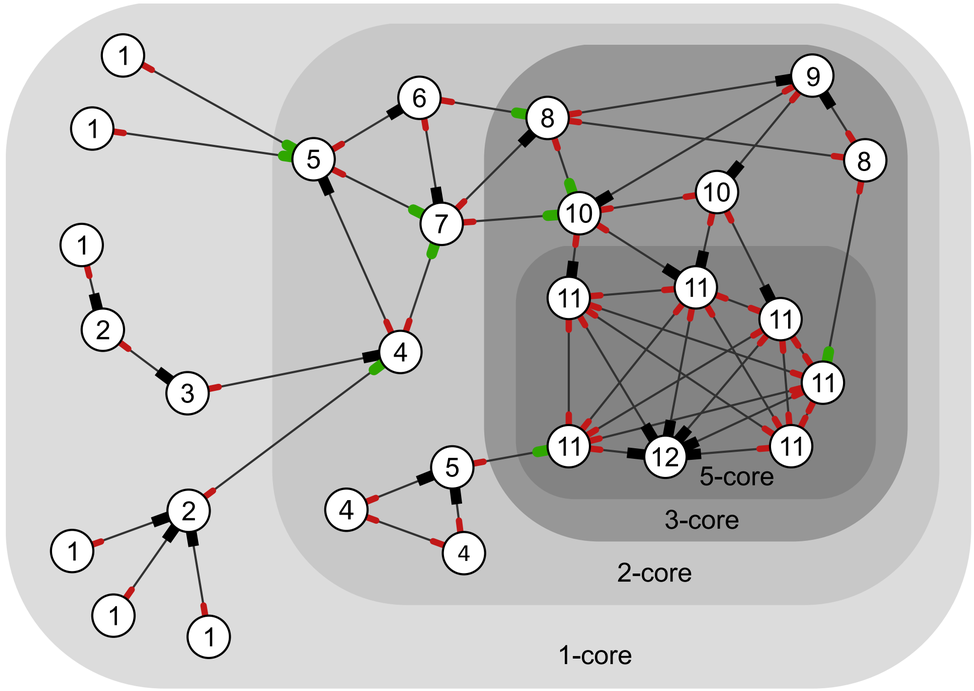
Onion decomposition: k-core decomposition with additional information about the positions of nodes within every k-shell (layers).

Information about layers is obtained from the k-core decomposition with minimal additional computational cost.

Stub matching scheme

- ▷ One type of nodes per layer
- ▷ Three types of stubs (red, green, black)
- ▷ Rules:

1. Allowed links: red–red, red–green, red–black
2. Nodes in layer ℓ and shell k must
   (a) have exactly k links to nodes in layers $\ell' \geq \ell$ (if layer ℓ is the first layer of the k-shell).
   (b) have at least $k+1$ links to nodes in layers $\ell' \geq \ell - 1$ and at most k links to nodes in layers $\ell' \geq \ell$ (if it is not in the first layer of the k-shell).



Mesoscopic level: The k-core/onion decomposition

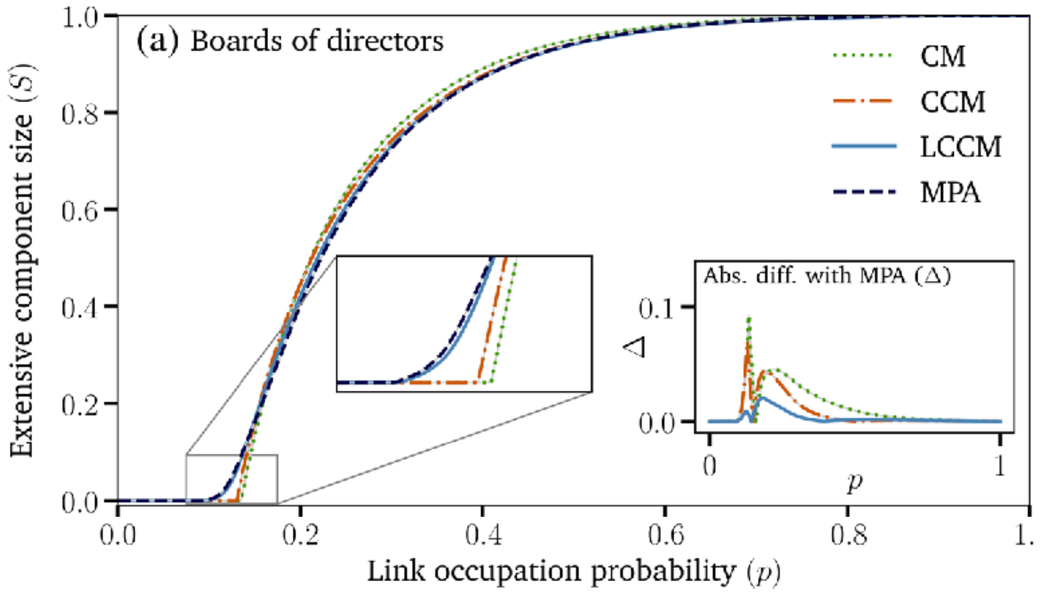
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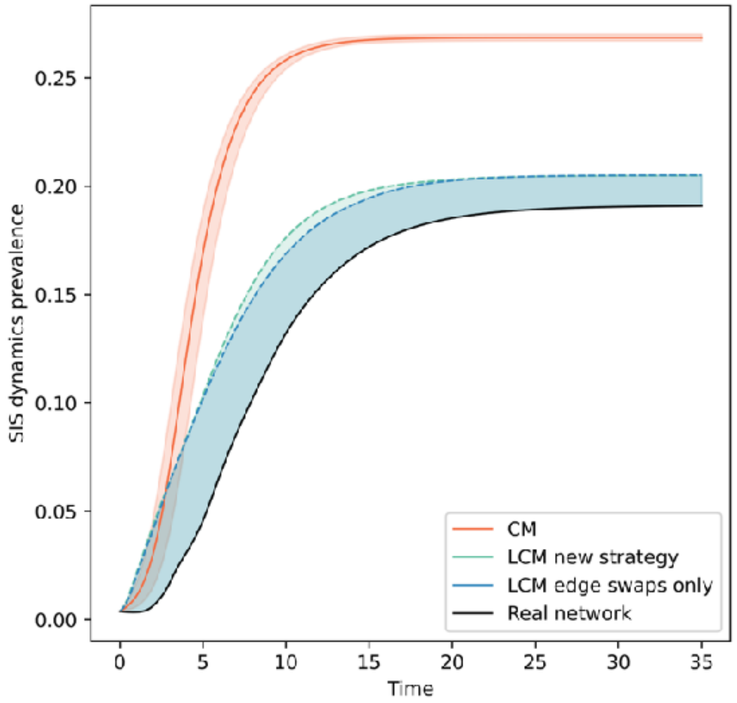
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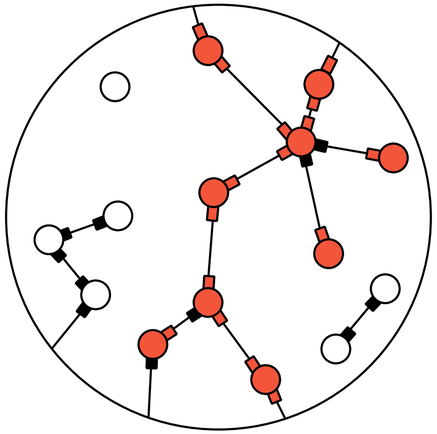
Macroscopic level: Connectivity

Preprocessing: Identify links that lead to the largest connected component and tag them as red; all remaining stubs are tags as black.

Stub matching scheme

- One type of nodes
- Two types of stubs (red, black)
- Rules:

1. Allowed links: red-red, red–black
2. Black stubs connect to any other stubs, but only towards nodes with excess red degree 0.
3. Red stubs connect to any other stubs, but only towards nodes with excess red degree at least 1.



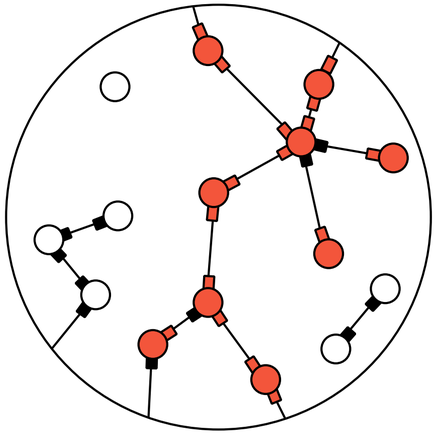
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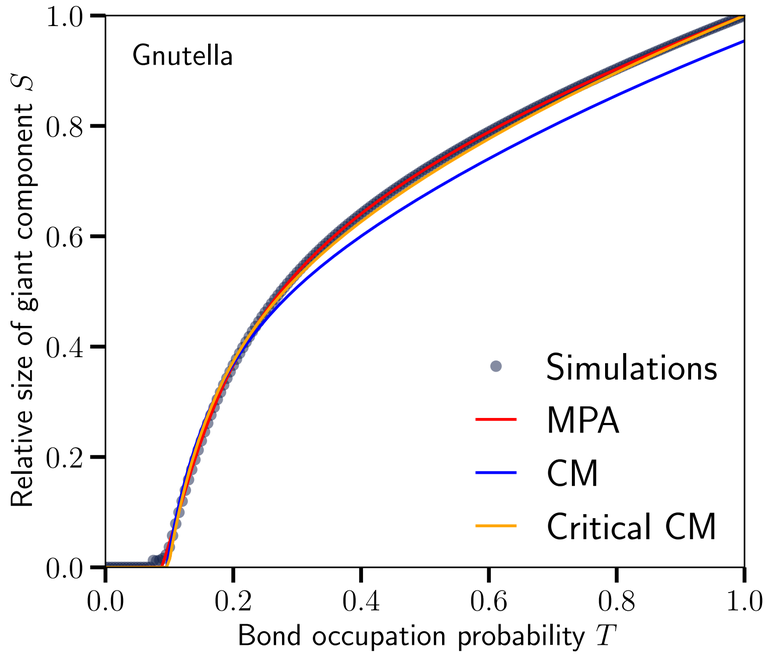
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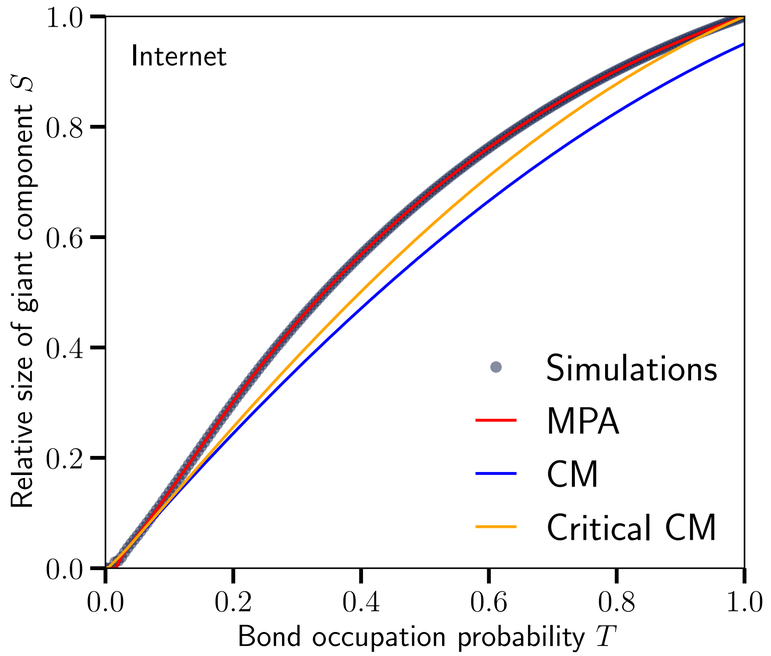
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Main takeaways

- ▷ Local connection rules can enforce strict global topological features.
- ▷ These rules can be leveraged to design
  - new mathematical frameworks (ex.: ODEs, PGFs)
  - new sampling algorithms (ex.: edge swapping)

Multi-scale structure and topological anomaly detection via a new network statistic: The onion decomposition

Laurent Hébert-Dufresne[1], Joshua A. Grochow[1] & Antoine Allard[2]

Sci. Rep. 6, 31708 (2016)

Percolation and the Effective Structure of Complex Networks

Antoine Allard[1,2] and Laurent Hébert-Dufresne[3,1]

Phys. Rev. X 9, 011023 (2019)

Modeling critical connectivity constraints in random and empirical networks

Laurent Hébert-Dufresne,[1,2,3] Márton Pósfai,[4] and Antoine Allard[3,5,1]

arXiv:2307.03559



Laurent Hébert-Dufresne
University of Vermont



François Thibault
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Márton Pósfai
Central European University

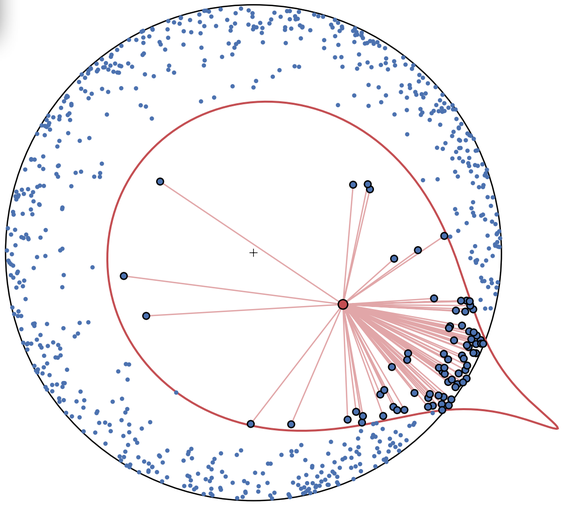


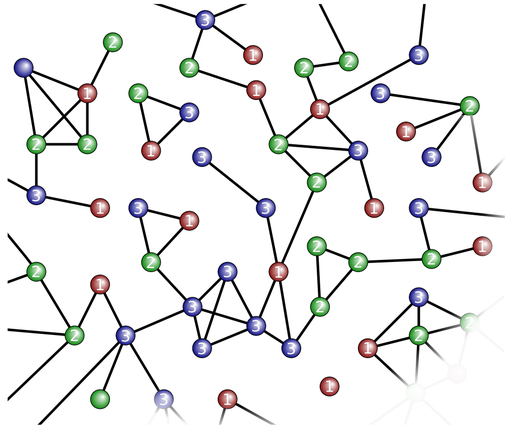
Joshua A. Grochow
University of Colorado, Boulder

Theoretical justifications exist for the use of "simple" models of complex networks and of the dynamics they support.

Simple models can reproduce nontrivial complex connections patterns observed in real networks.







slides, papers and code available at antoineallard.info