Simplicity meets complexity

Reproducing connectivity patterns in complex systems with simple models

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antoineallard.github.io

Complex systems

large number of elements + structured interactions

SCIENCE AND COMPLEXITY

By WARREN WEAVER

Rockefeller Foundation, New York City

American Scientist, 36, 536-544 (1948)

statistical methods hold the key. They are all problems which involve dealing simultaneously with a *sizable number of factors which are interrelated into an organic whole*. They are all, in the language here proposed, problems of *organized complexity*.

On what does the price of wheat depend? This too is a problem of organized complexity. A very substantial number of relevant variables is involved here, and they are all interrelated in a complicated, but nevertheless not in helter-skelter, fashion.

the whole is more than the sum of the parts

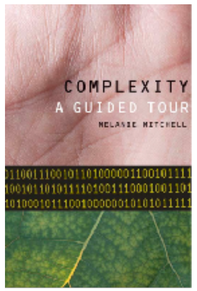
More Is Different

Broken symmetry and the nature of the hierarchical structure of science.

P. W. Anderson

Science, 177, 393-396 (1972)

erties of a few particles. Instead, at each level of complexity entirely new properties appear, and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other. That is, it



adaptation

All these systems adapt—that is, change their behavior to improve their chances of survival or success—through learning or evolutionary processes.

Outline

1. Are simple models enough to study complex systems/networks?

2. “Simple” ways to encode structural complexity

   (a) latent metric space

   (b) stub types

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2. “Simple” ways to encode structural complexity

   (a) latent metric space

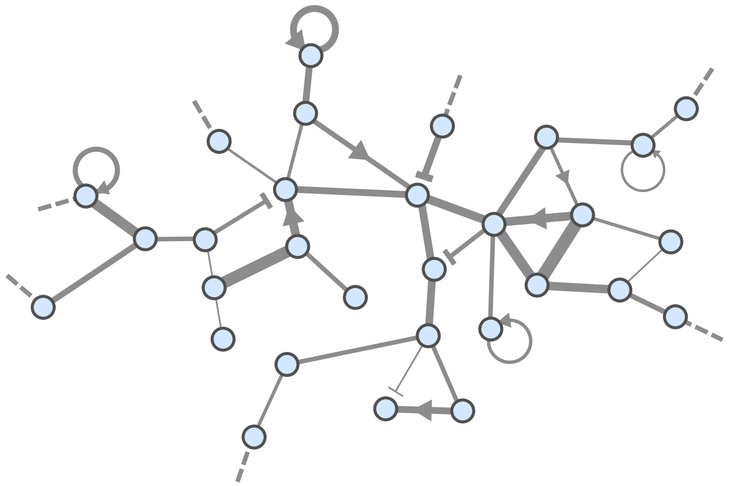
   (b) stub types



Modeling the dynamics of complex systems

Complex network

W

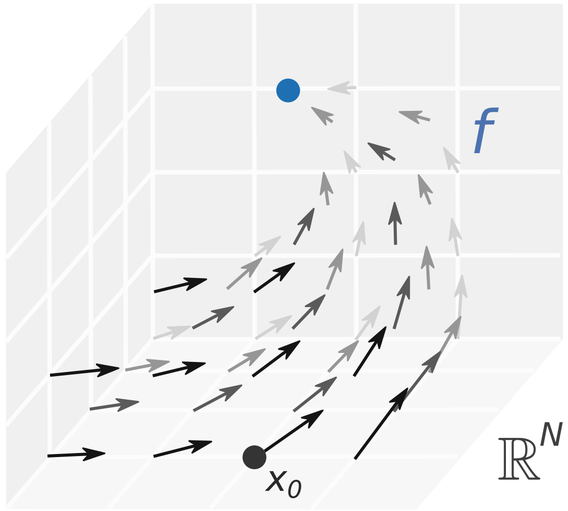


Vector field

f

High-dimensional dynamics

$\dot{x}=f(x;W)$



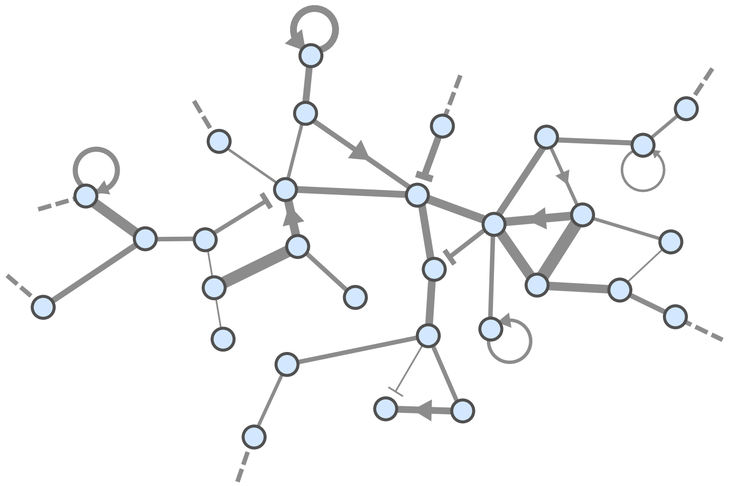
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Most mathematical models involve low-dimensional representations of the networks and/or dynamical observables.

Modeling the dynamics of complex systems

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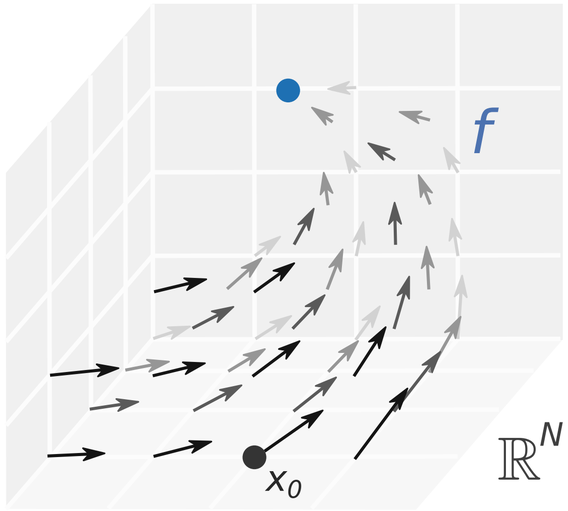


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Can we justify such a low-dimensional approach beyond mathematical and/or conceptual convenience?

The rank of adjacency matrices

Rank r of a matrix

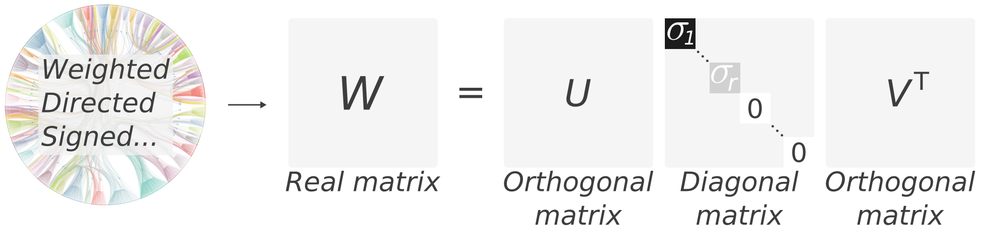
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Singular value decomposition (SVD)

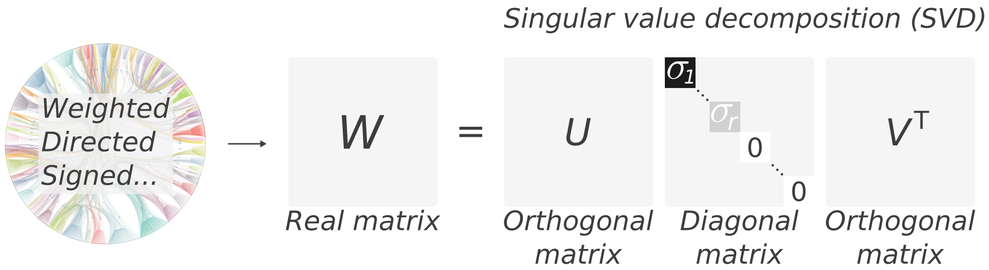


$$\mathbf{W} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\intercal = \sigma_1 \begin{bmatrix} | \\ \mathbf{u}_1 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_1^\intercal & — \end{bmatrix} + \sigma_2 \begin{bmatrix} | \\ \mathbf{u}_2 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_2^\intercal & — \end{bmatrix} + \sigma_3 \begin{bmatrix} | \\ \mathbf{u}_3 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_3^\intercal & — \end{bmatrix} + \dots$$

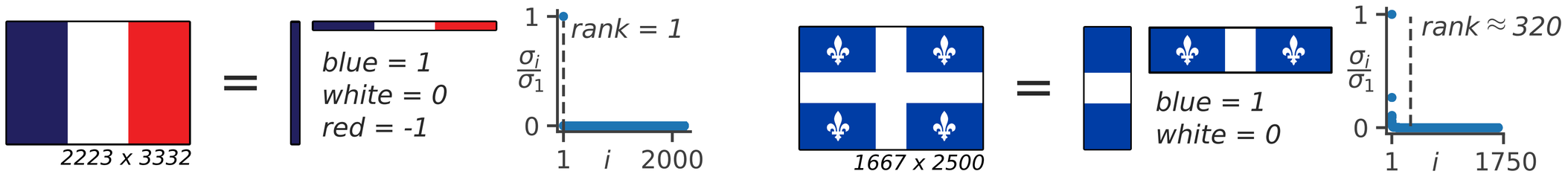
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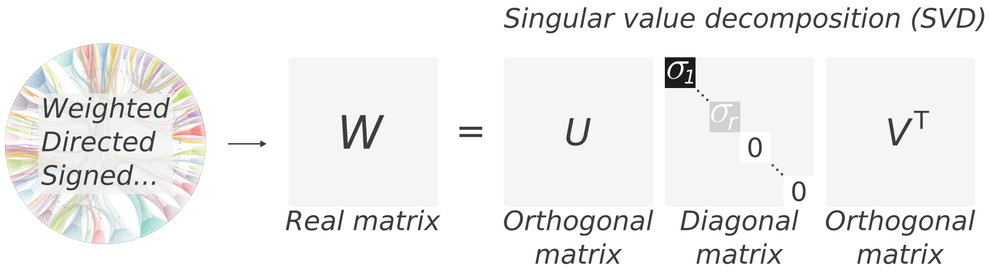
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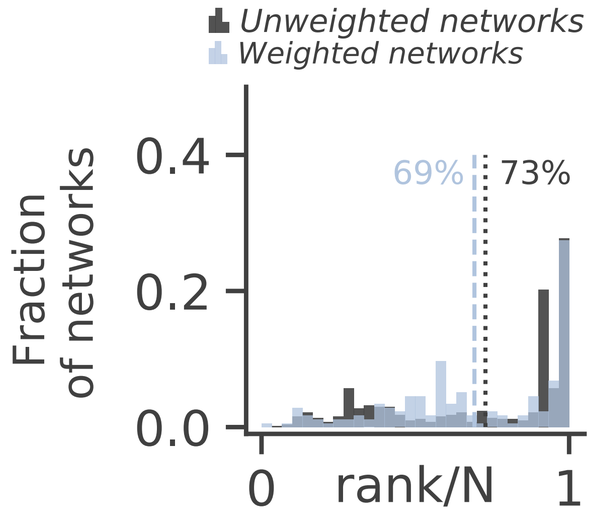
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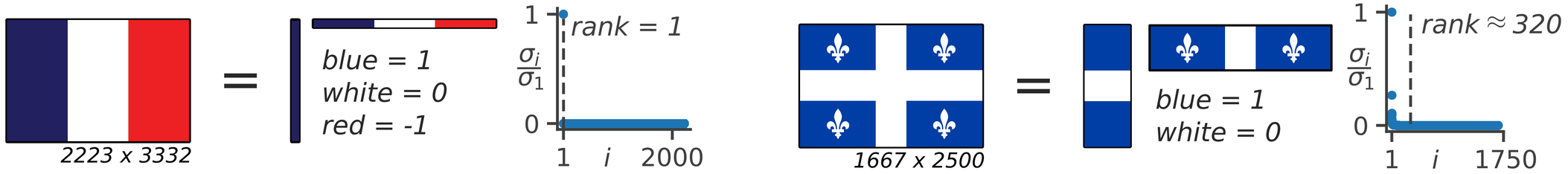


Full-rank (or almost full rank) adjacency matrices are very common!



Results for 679 empirical networks (502 unweighted networks and 177 weighted networks) downloaded from Netzschleuder.

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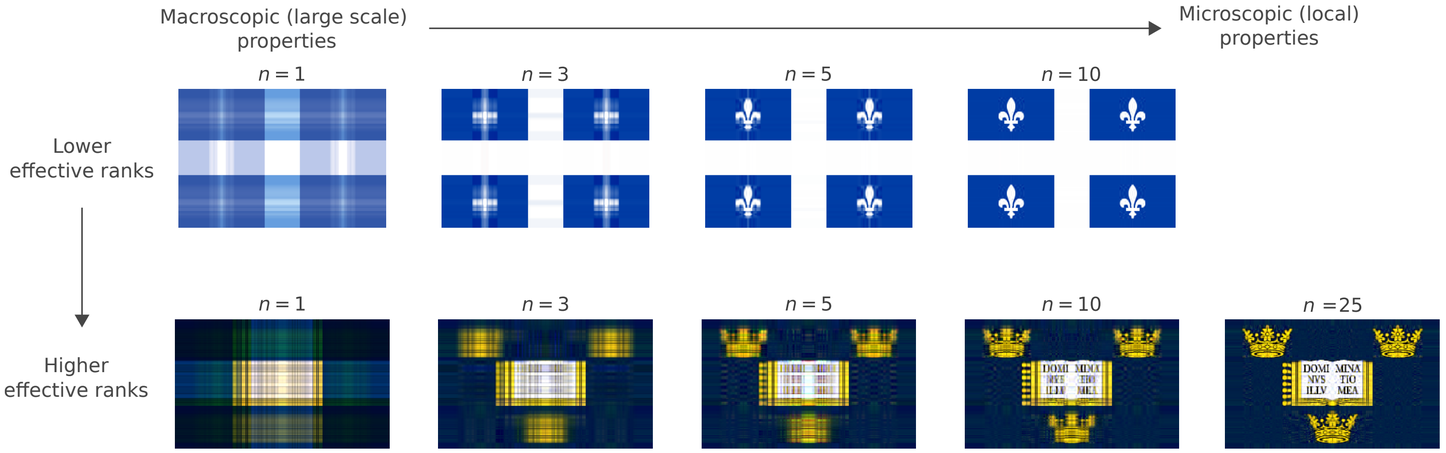


The effective ranks of adjacency matrices

$$\mathbf{W} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top \simeq \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$$

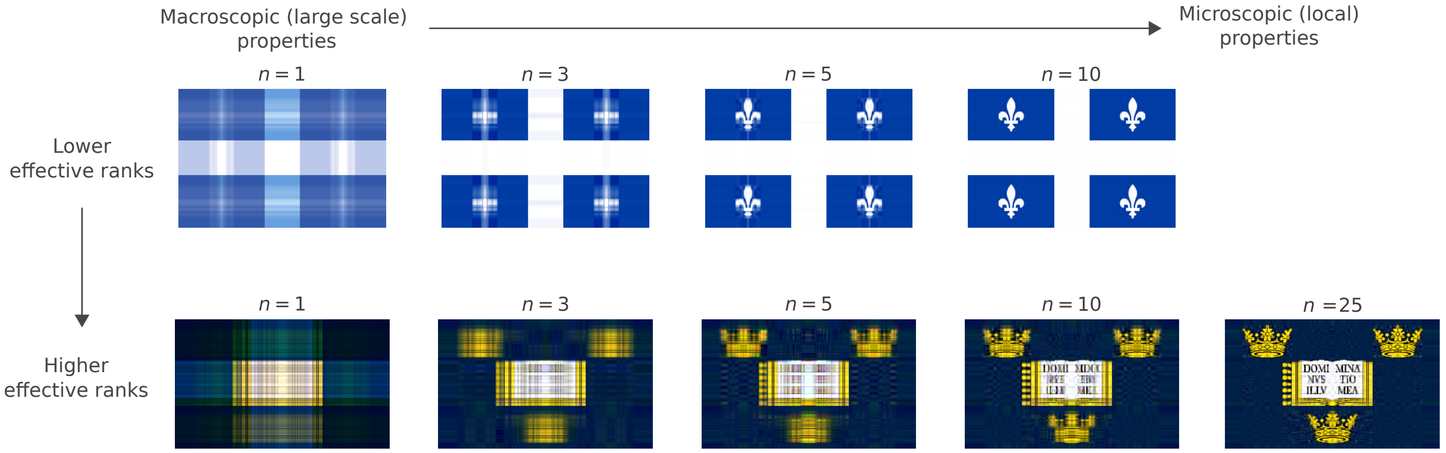
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Effective rank n of a matrix

- number of "significant" nonzero singular values
- term after which it is "reasonable" to truncate the sum

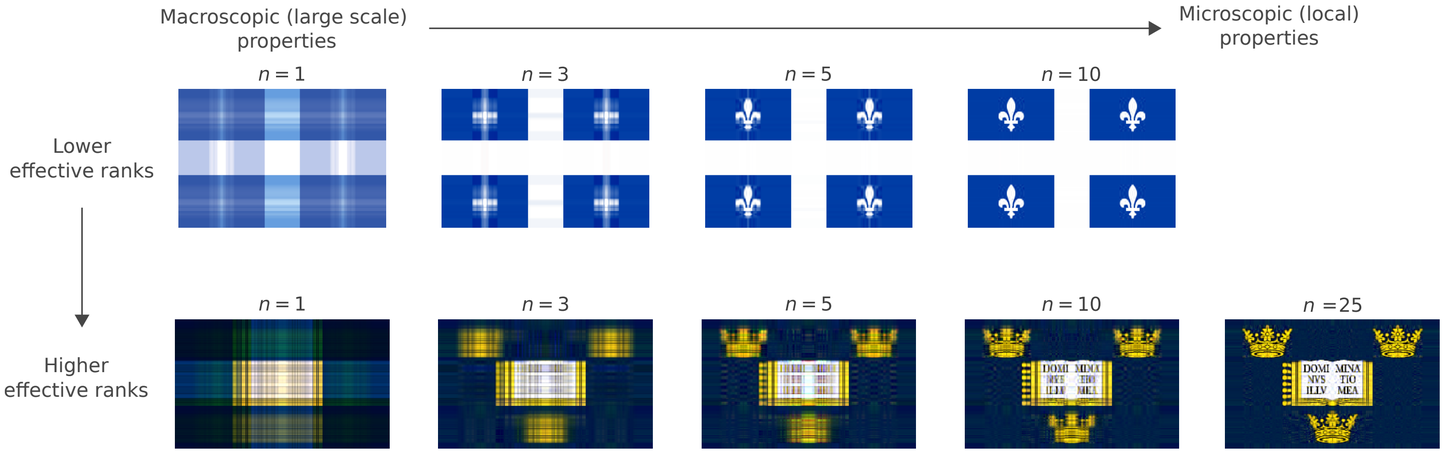
Abbreviation	Expression
srank	$\sum_{i=1}^{r} \sigma_i^2 / \sigma_1^2$
nrank	$\sum_{i=1}^{r} \sigma_i / \sigma_1$
energy	$\min\left[\arg\max_{\ell \in \{1,\ldots,N\}} \left(\sum_{i=1}^{\ell} \sigma_i^2 / \sum_{j=1}^{r} \sigma_j^2 > \tau\right)\right]$
elbow	$\frac{1}{\sqrt{2}} \arg\max_{i \in \{1,\ldots,N\}} \left\lvert \frac{i-1}{N-1} + \frac{\sigma_i - \sigma_N}{\sigma_1 - \sigma_N} - 1 \right\rvert - 1$
erank	$\exp\left\lvert -\sum_{i=1}^{r} \frac{\sigma_i}{\sum_{j=1}^{r} \sigma_j} \log \frac{\sigma_i}{\sum_{j=1}^{r} \sigma_j} \right\rvert$
thrank	$\#\left\{\sigma_i \mid i \in \{1, \ldots, N\} \text{ and } \sigma_i > \frac{4\sigma_{\text{med}}}{\sqrt{3}\mu_{\text{med}}}\right\}$
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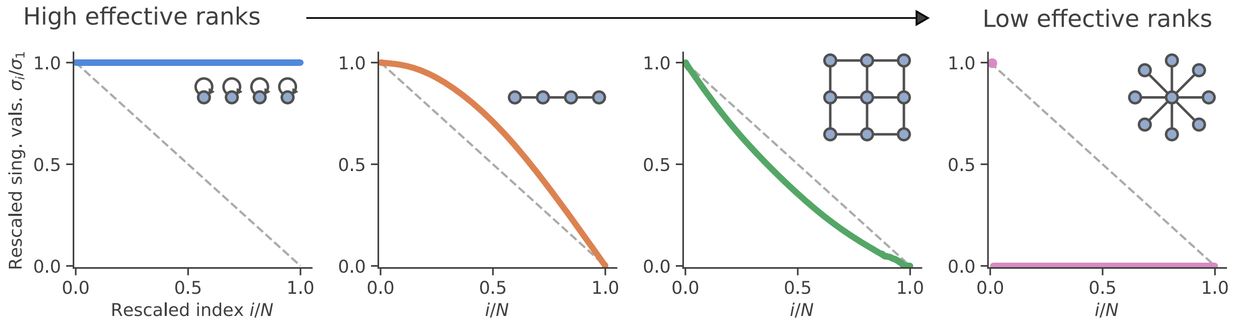
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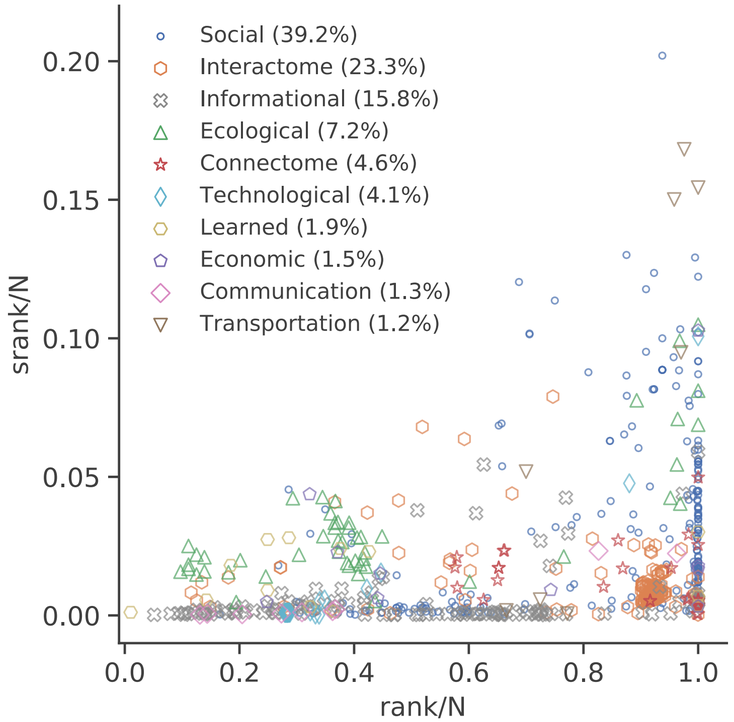
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The singular values (and the effective rank) encode information about the network's topology.



The effective ranks of adjacency matrices

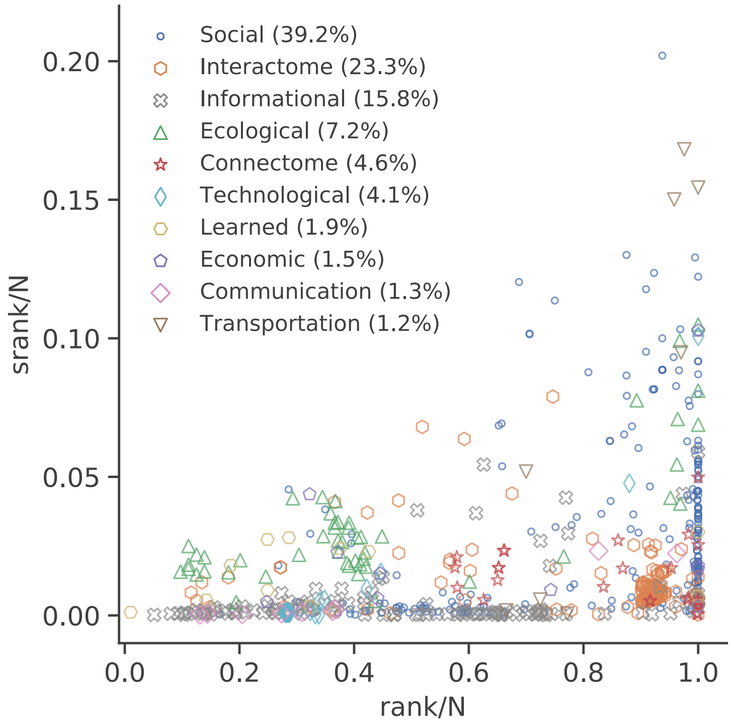
Many empirical networks appear to have a low effective rank!



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But what does “low” mean?

A *workable* definition of "low" effective rank

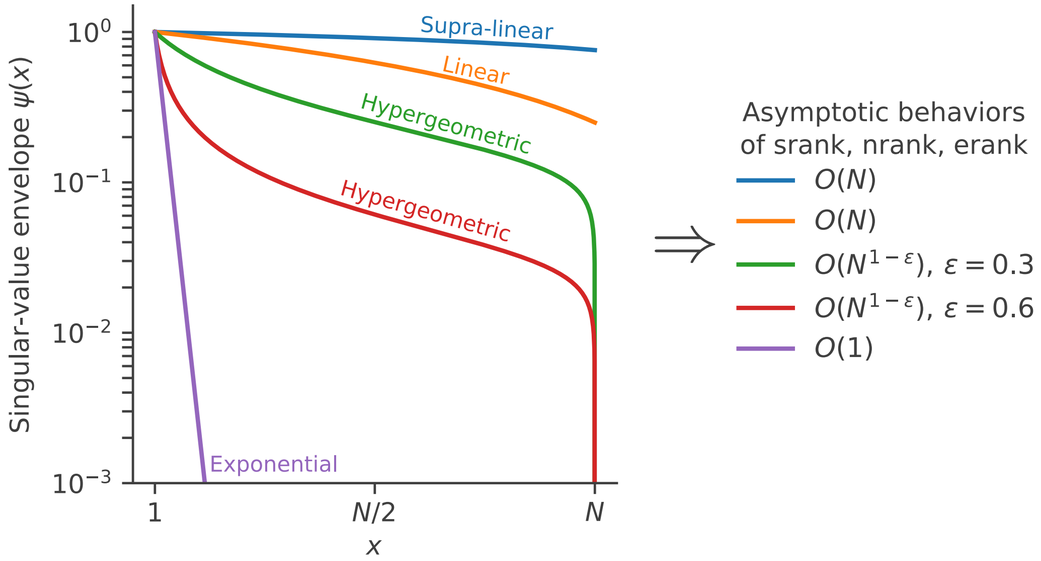
Hint: the rapid decrease of the dominant singular values of the adjacency matrix implies a low effective rank

- low effective rank? $\Rightarrow$ effective rank scales at most sublinearly as the number of nodes, N, goes to infitnity ($N^{1-\varepsilon}$ with $\varepsilon \in (0, 1]$)

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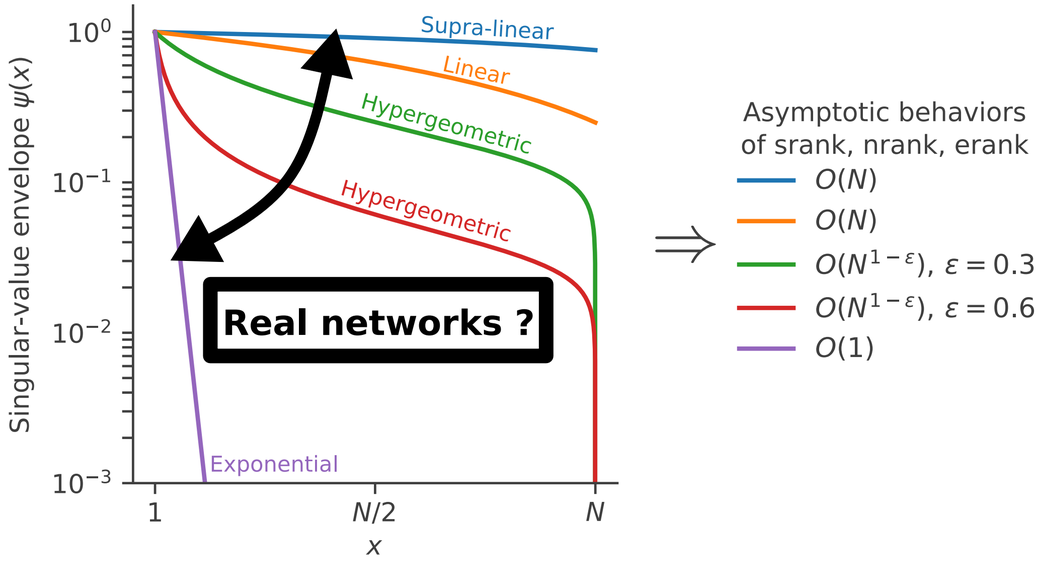
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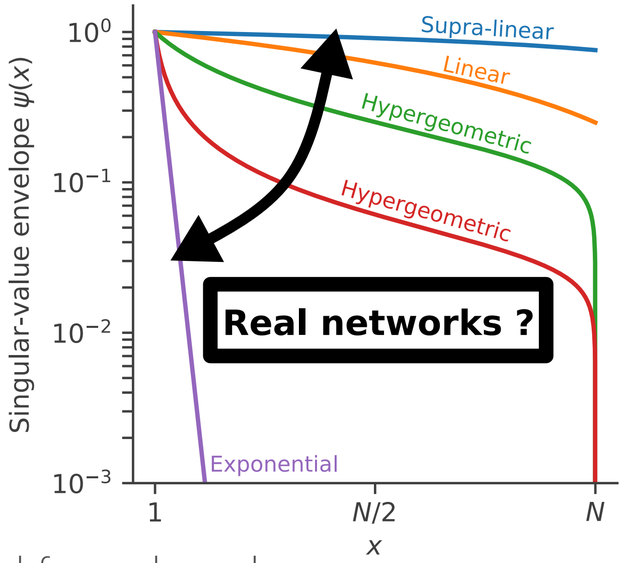
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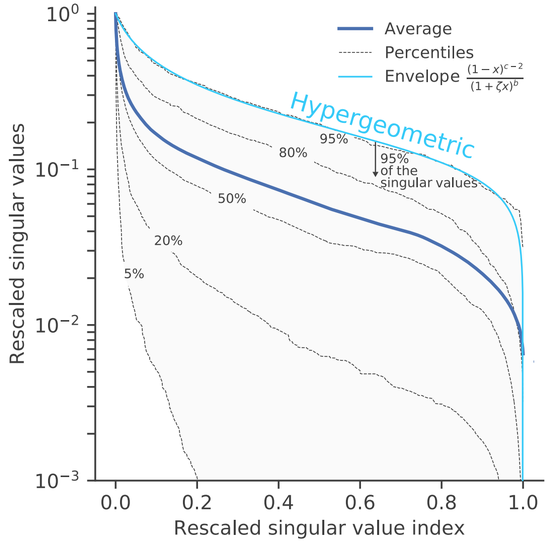


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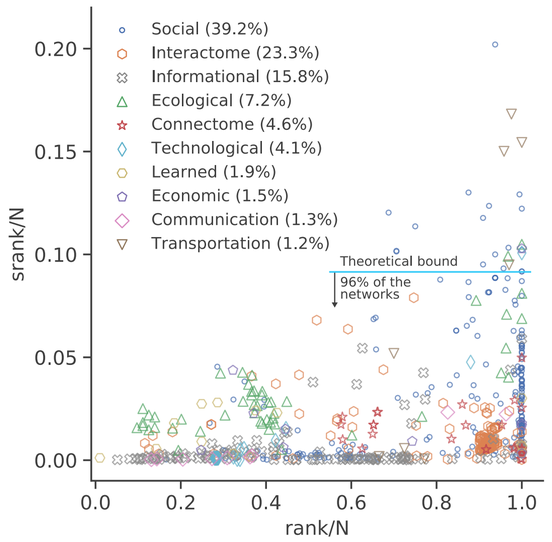
Asymptotic behaviors of srank, nrank, erank

- $O(N)$
- $O(N)$
- $O(N^{1-\varepsilon})$, $\varepsilon = 0.3$
- $O(N^{1-\varepsilon})$, $\varepsilon = 0.6$
- $O(1)$

The singular values of approx. 96% of considered networks are bounded from above by an hypergeometric envelope ⇒ sublinear effective ranks!



⇒



Workable definition of low effective rank: ∼10% of the number of nodes N

Approx. 96% of the 679 networks qualify for having a low effective rank!

The impact of effective low rank on the dynamics

Many dynamics on networks have the form

$$\dot{\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$$

with $\boldsymbol{x} \in \mathbb{R}^N$.

Examples:

- SIS (mean-field) : $\dot{x}_i = -d_i x_i + \gamma(1 - x_i)\, y_i$
- Wilson-Cowan: $\dot{x}_i = -d_i x_i + (1 - a x_i) \frac{1}{1+e^{-b(\gamma\, y_i - c)}}$
- Recurrent Neural Networks (RNN): $\dot{x}_i = -d_i x_i + \tanh(\gamma y_i + c_i)$
- Kuramoto-Sakaguchi: $\dot{z}_j = \mathrm{i}\omega_j z_j + \gamma\, e^{-\mathrm{i}\alpha}\, y_j - \gamma\, e^{\mathrm{i}\alpha}\, z_j^2\, \bar{y}_j$ with $z_j = e^{\mathrm{i}\theta_j}$
- Population dynamics: $\dot{x}_i = -d x_i + \gamma x_i y_i$ (Lotka-Volterra)
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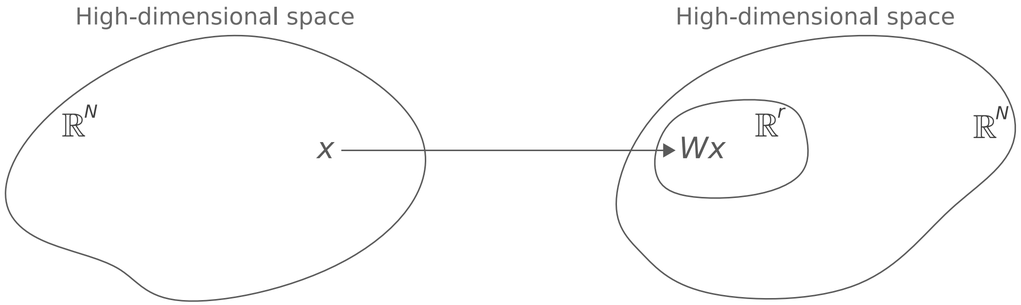
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Low rank $\mathbf{W} \Rightarrow \mathbf{W}\boldsymbol{x}$ belongs to a low-dimensional subspace



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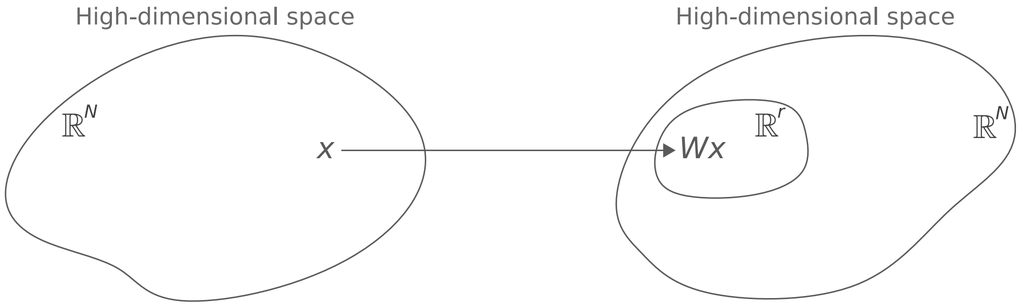
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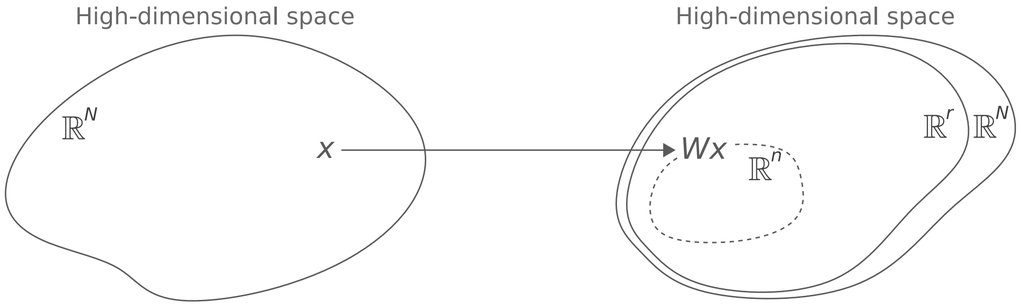
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Low *effective* rank $\mathbf{W} \Rightarrow \mathbf{W}\boldsymbol{x}$ belongs to an *effectively* low-dimensional subspace



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- Kuramoto-Sakaguchi: $\dot{z}_j = \mathrm{i}\omega_j z_j + \gamma\, e^{-\mathrm{i}\alpha}\, y_j - \gamma\, e^{\mathrm{i}\alpha}\, z_j^2\, \bar{y}_j$ with $z_j = e^{\mathrm{i}\theta_j}$
- Population dynamics: $\dot{x}_i = -d x_i + \gamma x_i y_i$ (Lotka-Volterra)
  $\dot{x}_i = -d x_i - s x_i^2 + \gamma \frac{x_i y_i}{\alpha + y_i}$
  $\dot{x}_i = a - d\, x_i + b x_i^2 - c x_i^3 + \gamma\, x_i\, y_i$

for $i, j \in \{1, ..., N\}$ and $y_i = \sum_{j=1}^{N} W_{ij} x_j$.

Original dynamics

$$\dot{\boldsymbol{x}} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x})$$

Reduced dynamics (with $\boldsymbol{X} = \mathbf{M}\boldsymbol{x}$)

$$\dot{\boldsymbol{X}} = \mathbf{M}\mathbf{g}(\mathbf{M}^{+}\mathbf{X}, \mathbf{W}\mathbf{M}^{+}\mathbf{X})$$

where n is the dimension of the reduced system
$\mathbf{M}^{+}$ denotes the pseudoinverse of $\mathbf{M}$
$\mathbf{M} = \mathbf{V}_n^{\top}$ is n-truncated right singular vector matrix

The alignment error is

$$\mathcal{E}(x) \leq \frac{1}{\sqrt{n}} \left[\|\mathbf{V}_n^{\top} \mathbf{J}'_x (\mathbf{I} - \mathbf{V}_n \mathbf{V}_n^{\top}) x\| + \sigma_{n+1} \|\mathbf{V}_n^{\top} \mathbf{J}'_y\|_2 \|x\| \right]$$

where $\mathbf{J}'_x, \mathbf{J}'_y$ are Jacobian matrices

The impact of effective low rank on the dynamics

Many dynamics on networks have the form

$$\dot{\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$$

with $\boldsymbol{x} \in \mathbb{R}^N$.

Examples:

- SIS (mean-field) : $\dot{x}_i = -d_i x_i + \gamma(1 - x_i)\, y_i$
- Wilson-Cowan: $\dot{x}_i = -d_i x_i + (1 - a x_i)\frac{1}{1+e^{-b(\gamma\, y_i - c)}}$
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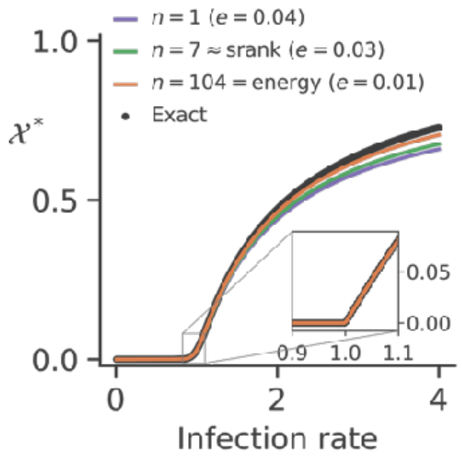
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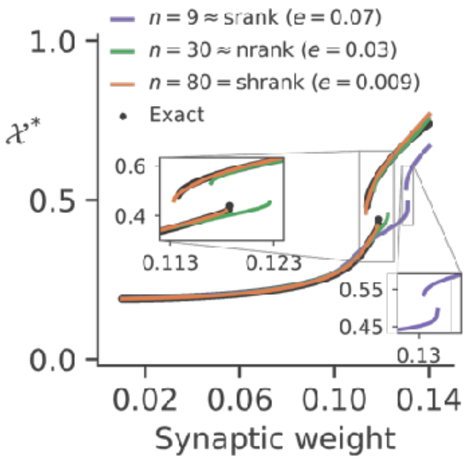
$$\mathcal{E}(x) \leq \frac{1}{\sqrt{n}} \left[\|\mathbf{V}_n^{\top} \mathbf{J}'_x (\mathbf{I} - \mathbf{V}_n \mathbf{V}_n^{\top}) x\| + \sigma_{n+1} \|\mathbf{V}_n^{\top} \mathbf{J}'_y\|_2 \|x\| \right]$$

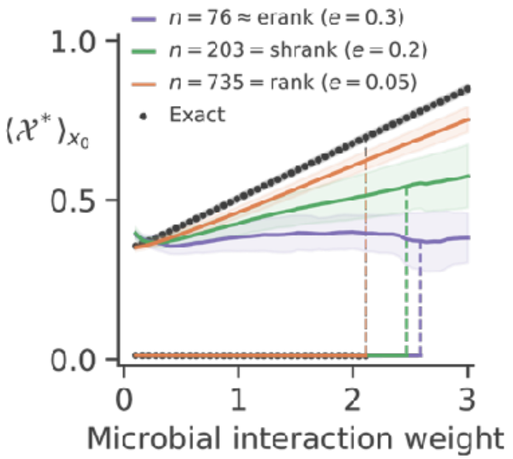
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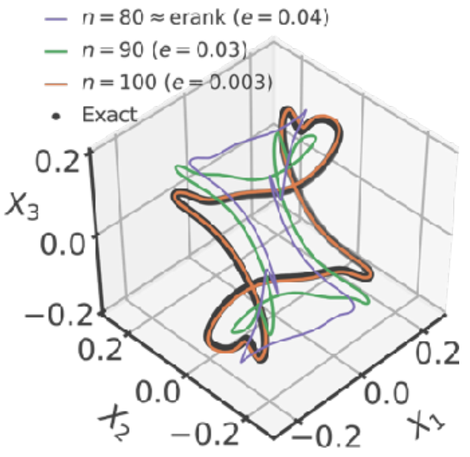
Rapid singular value decrease can induce
rapid alignment error decrease!

Reproduction of the dynamics with increasing accuracy

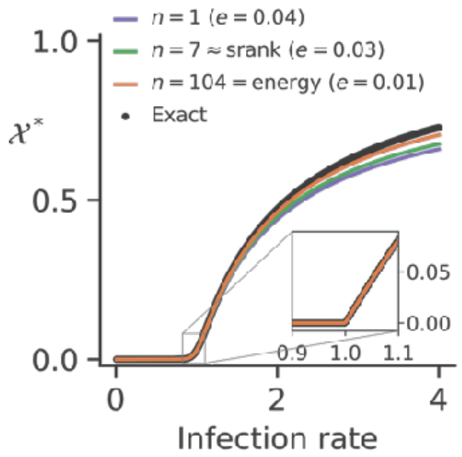


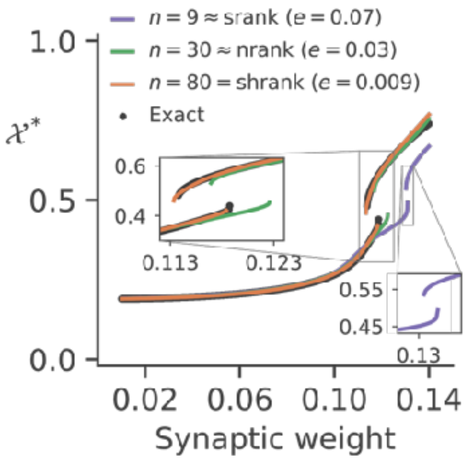


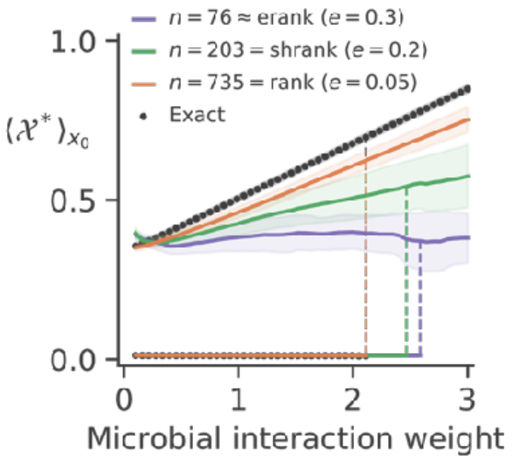


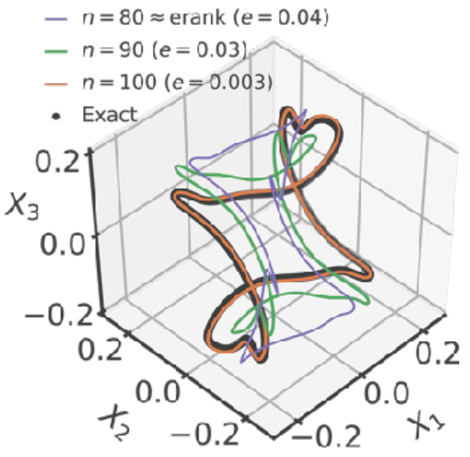


Reproduction of the dynamics with increasing accuracy









Emergence of higher-order interactions

Original Quenched mean-field SIS dynamics

$$\dot{x}_i = -\alpha x_i + \beta(1-x_i)\sum_{j=1}^{N} W_{ij}x_j, \quad i \in \{1,...,N\}$$

Reduced dynamics

$$\dot{X}_\mu = -\alpha X_\mu + \beta\sum_{\nu=1}^{n}\mathcal{W}_{\mu\nu}X_\nu - \beta\sum_{\nu,\tau=1}^{n}\mathcal{W}^{(3)}_{\mu\nu\tau}X_\nu X_\tau, \quad \mu \in \{1,...,n\}$$

Original Kuramoto-Sakaguchi dynamics

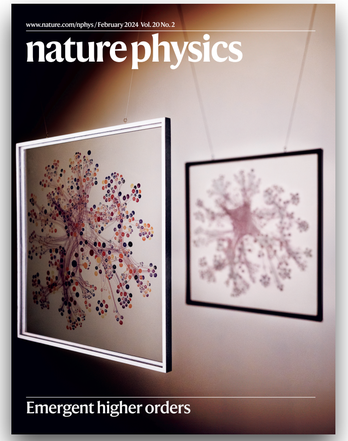
$$\dot{z}_j = i\omega_j z_j + \sum_{k=1}^{N} W_{jk}[z_k e^{-i\alpha} - z_j^2 \bar{z}_k e^{i\alpha}], \quad j \in \{1,...,N\}$$

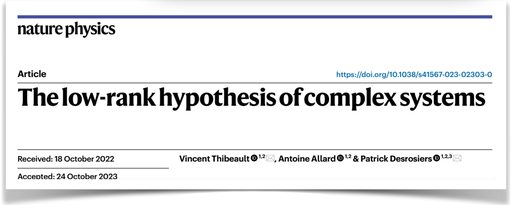
Reduced dynamics

$$\dot{Z}_\mu = i\sum_{\nu=1}^{n}\Omega_{\mu\nu}Z_\nu + \sum_{\nu=1}^{n}\mathcal{W}^{(2)}_{\mu\nu}Z_\nu e^{-i\alpha} - \sum_{\alpha,\beta,\gamma=1}^{n}\mathcal{W}^{(4)}_{\mu\alpha\beta\gamma}Z_\alpha Z_\beta \bar{Z}_\gamma e^{i\alpha}$$

Main takeaways

- The rapid decrease of the singular values of adjacency matrices (i.e. low effective rank) offers a *justification* for low-dimensional mathematical models beyond mathematical and/or conceptual convenience.
- A large proportion of real networks can be considered as having a low effective rank.
- The higher-order interactions observed in some systems could be a byproduct of a low-dimensional representation used to analyze them.




nature physics

Article

https://doi.org/10.1038/s41567-023-02303-0

The low-rank hypothesis of complex systems

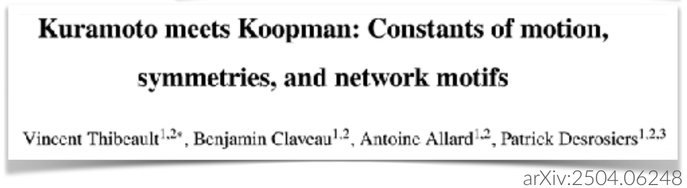
Received: 18 October 2022

Accepted: 24 October 2023

Vincent Thibeault, Antoine Allard & Patrick Desrosiers

Challenges and open questions

- Could we measure the effective dimension independently?
- Could we design a random graph model based on observed singular values (singular vectors)?
- Are some of the higher-order interactions inferred from time series artifacts of coarse-grained observations?
- Could we designed more interpretable observables, perhaps nonlinear ones?


Kuramoto meets Koopman: Constants of motion, symmetries, and network motifs

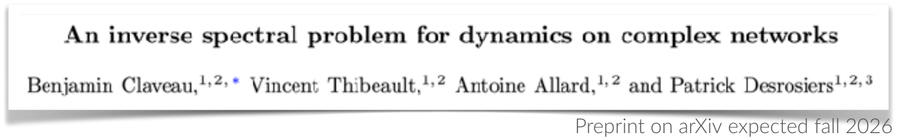
Vincent Thibeault[1,2*], Benjamin Claveau[1,2], Antoine Allard[1,2], Patrick Desrosiers[1,2,3]

arXiv:2504.06248


Operator-theoretic approach to the partial integration of randomly coupled phase oscillators

Vincent Thibeault,[1,2,*] Benjamin Claveau,[1,2] Antoine Allard,[1,2] and Patrick Desrosiers[1,2,3]

arXiv:2607.02617


An inverse spectral problem for dynamics on complex networks

Benjamin Claveau,[1,2,*] Vincent Thibeault,[1,2] Antoine Allard,[1,2] and Patrick Desrosiers[1,2,3]

Preprint on arXiv expected fall 2026



Vincent Thibeault
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Patrick Desrosiers
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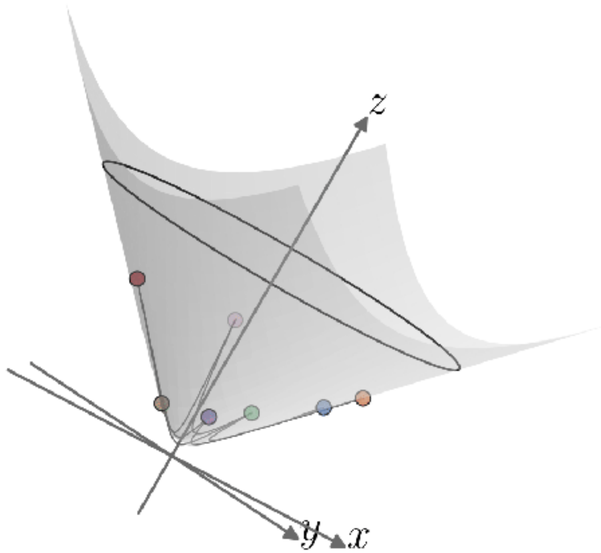
Outline

1. Are simple models enough to study complex systems/networks?

2. **“Simple” ways to encode structural complexity**

   (a) **latent metric space**

   (b) stub types



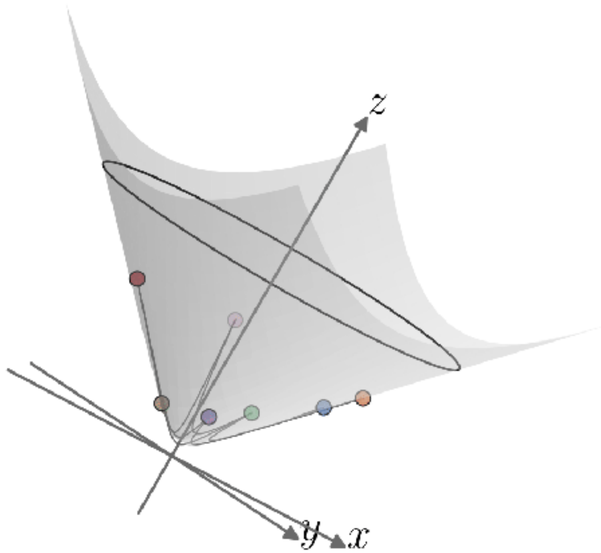
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Maximally random graph ensembles

The probability, $P(\mathbb{A})$, for a $N \times N$ adjacency matrix $\mathbb{A} = \{a_{ij}\} \in [0,1]^{\binom{N}{2}}$ that maximizes the entropy subjected to the L constraints ($l = 1, 2, \ldots, L$)

$$S(\{\mathbb{A}\}) = -\sum_{\mathbb{A}} P(\mathbb{A}) \ln P(\mathbb{A}) \qquad \bar{F}_l = \sum_{\mathbb{A}} F_l(\mathbb{A}) P(\mathbb{A})$$

is (α_l being the l-th Lagrange multiplier)

$$P(\mathbb{A}) \propto \exp\left(-\sum_{l=1}^{L} \alpha_l F_l(\mathbb{A})\right) .$$

Phys. Rev. E 70, 066117 (2004)
EPL 81, 28005 (2007)
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Example 1: fixing the expected number of edges

$$\bar{F}_1 = \sum_{i=1}^{N} \sum_{j=i+1}^{N} a_{ij} = M$$

yields the Bernouilli random graph ensemble

$$P(\mathbb{A}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} p^{a_{ij}} (1-p)^{1-a_{ij}} \quad \text{with} \quad p = \frac{1}{e^{\alpha_1} + 1} .$$

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Example 2: fixing the expected degree sequence

$$\bar{F}_l = \sum_{j=1}^{N} a_{lj} = k_l$$

for $l = 1, \ldots, N$ yields the soft configuration model

$$P(\mathbb{A}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} p_{ij}^{a_{ij}} (1-p_{ij})^{1-a_{ij}} \quad \text{with} \quad p_{ij} = \frac{1}{e^{-(\alpha_i+\alpha_j)} + 1} .$$

Redefining $\kappa_l = \sqrt{\langle\kappa\rangle N} e^{\alpha_l}$ for $l = 1, \ldots, N$ yields the Chung-Lu model

$$p_{ij} = \frac{1}{1 + \frac{\langle\kappa\rangle N}{\kappa_i \kappa_j}} \simeq \frac{\kappa_i \kappa_j}{\langle\kappa\rangle N} .$$

Studying the conditions for which p_{ij} can be factorized informs us on the degree-degree correlations observed in real networks.

Phys. Rev. E 70, 066117 (2004)
EPL 81, 28005 (2007)
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Phys. Rev. E 86, 026120 (2012)

Modeling clustering

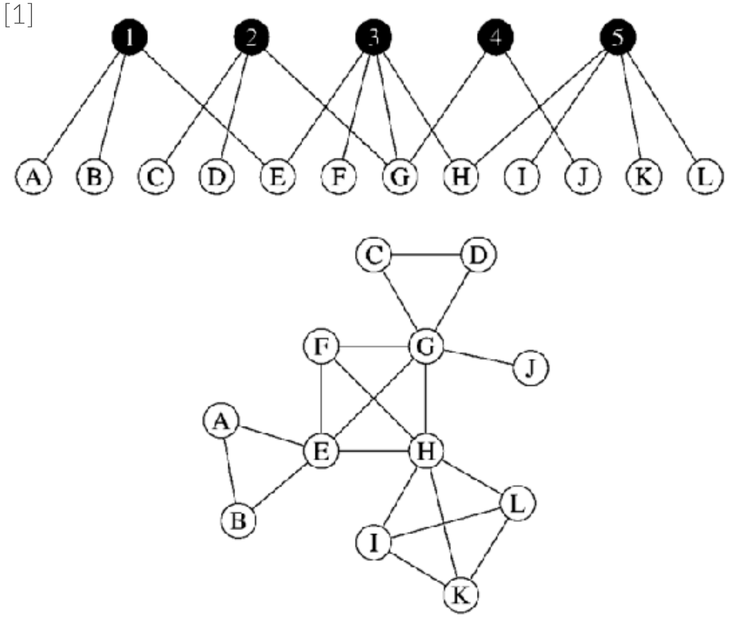
Tricky because clustering consists in three-node interactions while our mathematical tools rely on pairwise interactions either explicitly or implicitly.

Straightforward inclusion of triangles to the maximally random graph ensemble formalism yields unwanted behavior (ex.: triangle agglutination in the Strauss model [6]).

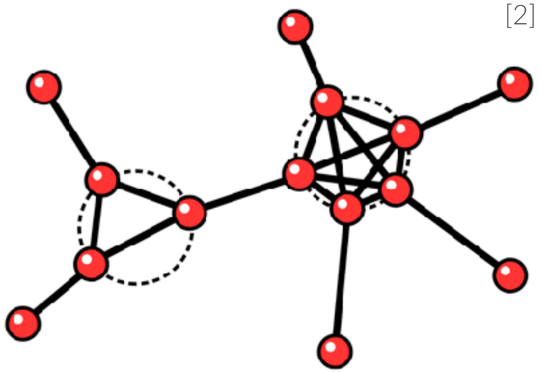
Most models therefore assume

- ▷ an underlying tree-like structure
- ▷ that the networks are dense

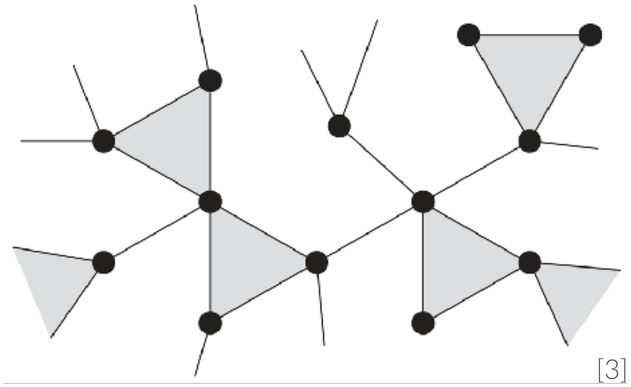
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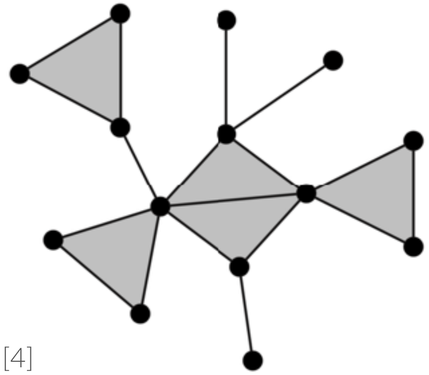
[2]



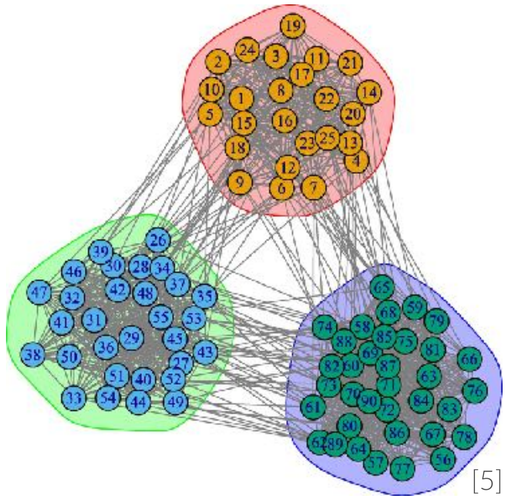
[3]



[4]



[5]



[1] Phys. Rev. E 68, 026121 (2003)
[2] Phys. Rev. E 80, 036107 (2009)
[3] Phys. Rev. Lett. 103, 058701 (2009)
[4] Phys. Rev. E 82, 066118 (2010)
[5] Appl. Netw. Sci. 4, 122 (2019)
[6] Phys. Rev. E 72, 026136 (2005)

Modeling clustering

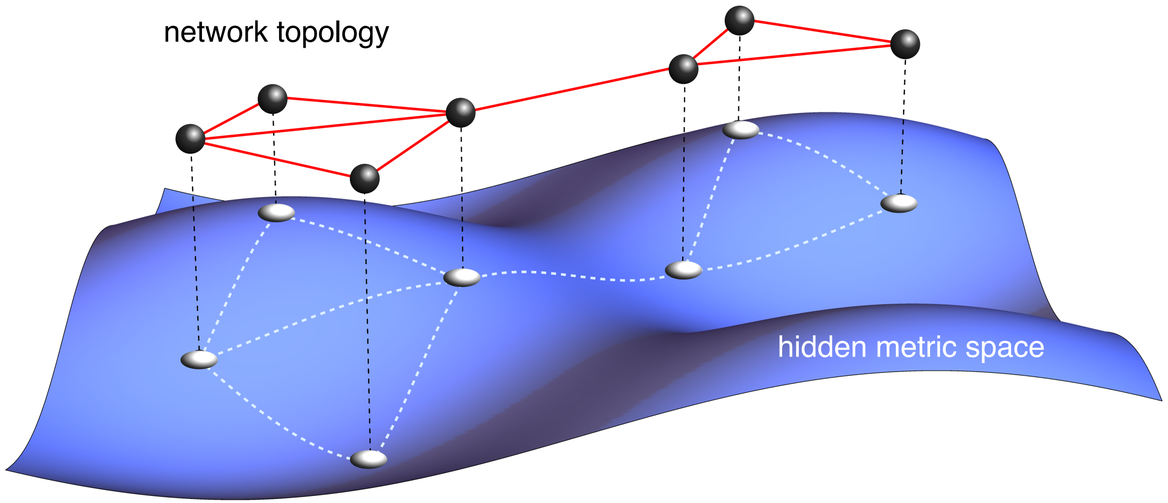
Idea: Assume that the vertices are embedded in a metric space and that any two vertices are connected with a probability that is a decreasing function of the distance between them.

Modeling clustering

Idea: Assume that the vertices are embedded in a metric space and that any two vertices are connected with a probability that is a decreasing function of the distance between them.

Properties of any metric space:

Identity of indiscernibles	$d(x,y) = 0 \quad \Leftrightarrow \quad x = y$
Non-negativity	$d(x,y) \geq 0$
Symmetry	$d(x,y) = d(y,x)$
Triangle inequality	$d(x,y) \leq d(x,z) + d(z,y)$



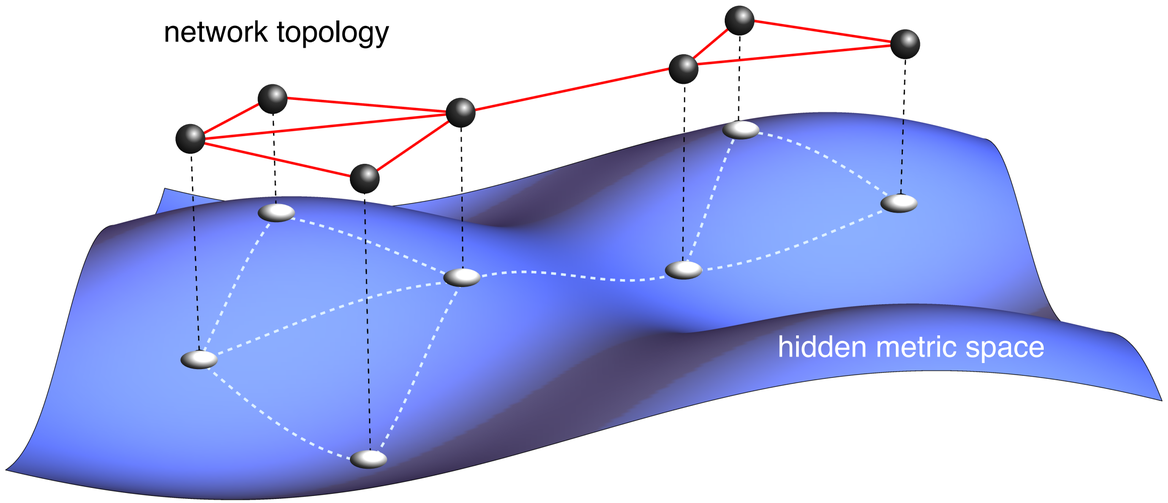
Courtesy of M. Boguñá

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Courtesy of M. Boguñá

Henceforth we consider that there is a distance matrix $\mathbb{X} = \{x_{ij}\}$ where x_{ij} is the distance between vertices i and j in the D-dimensional space.

Maximally random geometric graph ensembles

Example 3: fixing the expected number of edges and the expected total energy

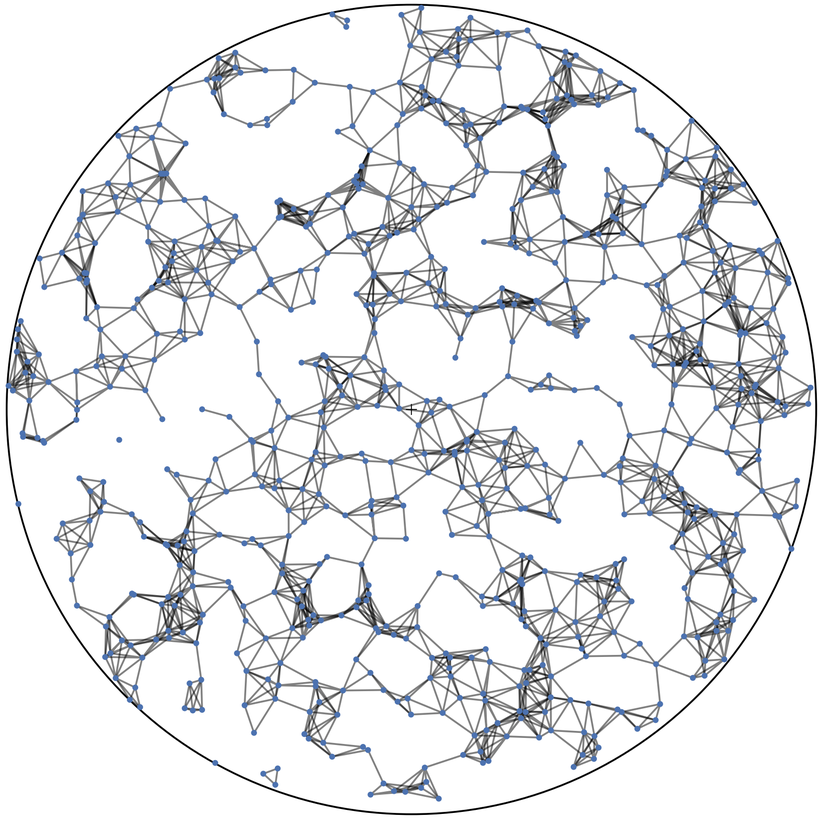
$$\bar{F}_1 = \sum_{i=1}^{N} \sum_{j=i+1}^{N} a_{ij} = M$$

$$\bar{F}_2 = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \varepsilon_{ij} a_{ij} = \sum_{i=1}^{N} \sum_{j=i+1}^{N} f(x_{ij}) a_{ij} = E$$

yields the homogeneous random geometric graph ensemble

$$P(\mathbb{A}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} p_{ij}^{a_{ij}} (1 - p_{ij})^{1 - a_{ij}} \quad \text{with} \quad p_{ij} = \frac{1}{e^{\beta(\varepsilon_{ij} - \mu)} + 1} \, .$$

The graphs will be sparse, highly clustered and small-world iif $f(x_{ij}) \sim \ln x_{ij}$ and $\beta \in [D, D + 2]$.



Maximally random geometric graph ensembles

Example 4: fixing the expected degree sequence and the expected total energy

$$\bar{F}_l = \sum_{j=1}^{N} a_{lj} = k_l \quad (l = 1, \ldots, N)$$

$$\bar{F}_{N+1} = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \varepsilon_{ij} a_{ij} = \sum_{i=1}^{N} \sum_{j=i+1}^{N} f(x_{ij}) a_{ij} = E$$

yields the heterogeneous random geometric graph ensemble

$$P(\mathbb{A}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} p_{ij}^{a_{ij}} (1 - p_{ij})^{1 - a_{ij}} \quad \text{with} \quad p_{ij} = \frac{1}{e^{\beta \varepsilon_{ij} - \alpha_i - \alpha_j} + 1} \, .$$

The graphs will be sparse, highly clustered, small-world and devoid of non-structural degree-degree correlation iif $f(x_{ij}) = \ln x_{ij}$ and $\beta \in [D, D+2]$[a]. Redefining $\alpha_l = -(\beta/D) \ln(\sqrt{\mu} \kappa_l)$ yields

$$p_{ij} = \frac{1}{e^{\beta(\varepsilon_{ij} - \mu)} + 1} \quad \text{with} \quad \varepsilon_{ij} = \ln \left(\frac{x_{ij}}{(\kappa_i \kappa_j)^{\frac{1}{D}}} \right) \, .$$

[a] No upper bound if expected degree sequence is scale-free.

Phys. Rev. E 80, 035101 (2009)
Phys. Rev. E 82, 036106 (2010)
Phys. Rev. Research 2, 023040 (2020)

Maximally random geometric graph ensembles

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yields the heterogeneous random geometric graph ensemble

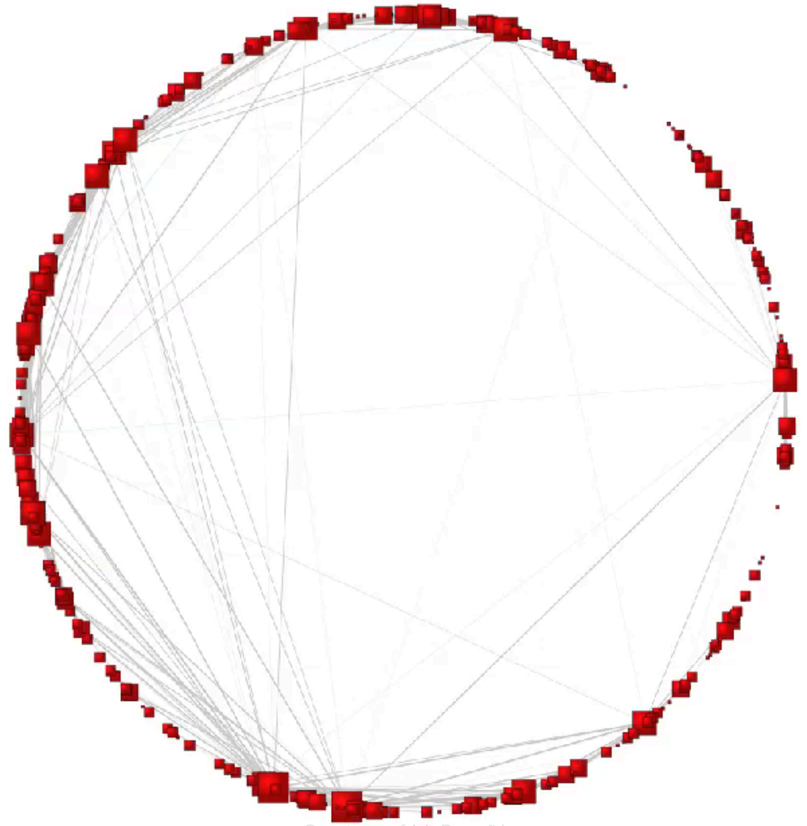
$$P(\mathbb{A}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} p_{ij}^{a_{ij}} (1 - p_{ij})^{1 - a_{ij}} \quad \text{with} \quad p_{ij} = \frac{1}{e^{\beta \varepsilon_{ij} - \alpha_i - \alpha_j} + 1} \, .$$

The graphs will be sparse, highly clustered, small-world and devoid of non-structural degree-degree correlation iif $f(x_{ij}) = \ln x_{ij}$ and $\beta \in [D, D+2]$[a]. Redefining $\alpha_l = -(\beta/D) \ln(\sqrt{\mu} \kappa_l)$ yields

$$p_{ij} = \frac{1}{e^{\beta(\varepsilon_{ij} - \mu)} + 1} \quad \text{with} \quad \varepsilon_{ij} = \ln \left(\frac{x_{ij}}{(\kappa_i \kappa_j)^{\frac{1}{D}}} \right) \, .$$

[a] No upper bound if expected degree sequence is scale-free.

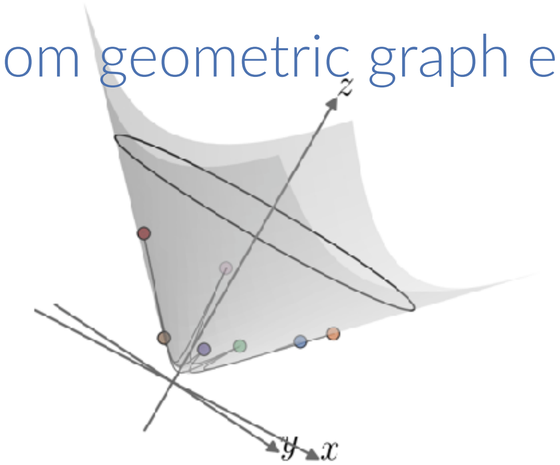
When the geometry is a D-dimensional sphere, $\mathbb{S}^D$ the model can be mapped to a purely geometric model in hyperbolic space $\mathbb{H}^{D+1}$.



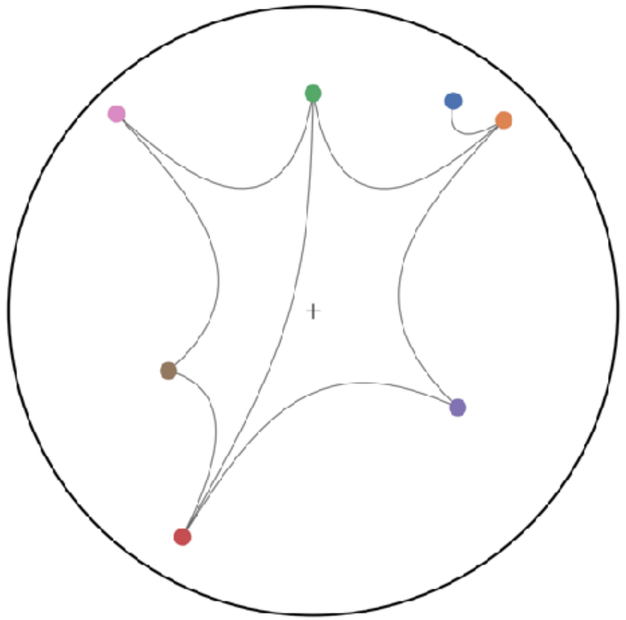
Courtesy of M. Boguñá

Phys. Rev. E 80, 035101 (2009)
Phys. Rev. E 82, 036106 (2010)
Phys. Rev. Research 2, 023040 (2020)

Maximally random geometric graph ensembles

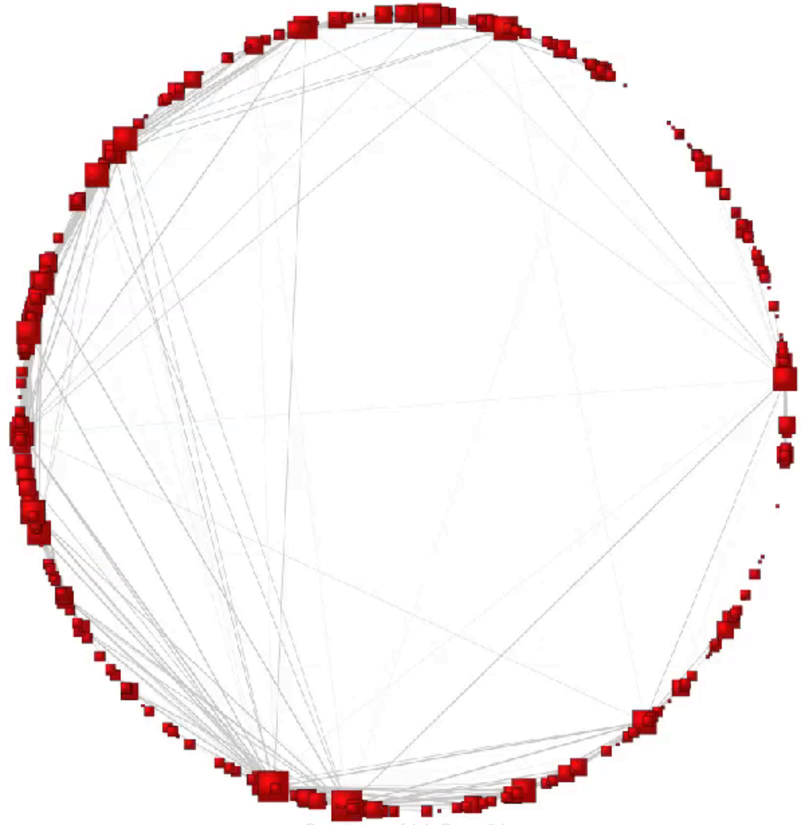


hyperboloid in $\mathbb{R}^{2,1}$



hyperbolic disk (r, θ)

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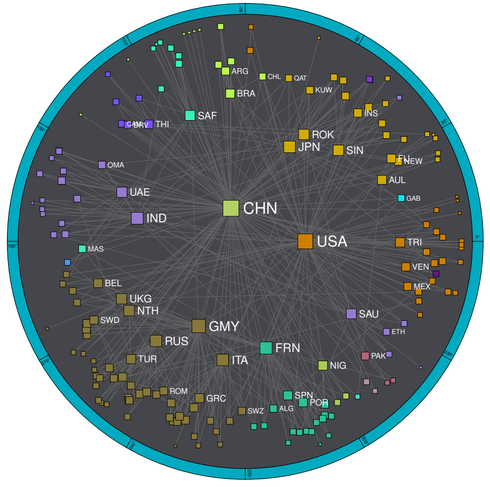


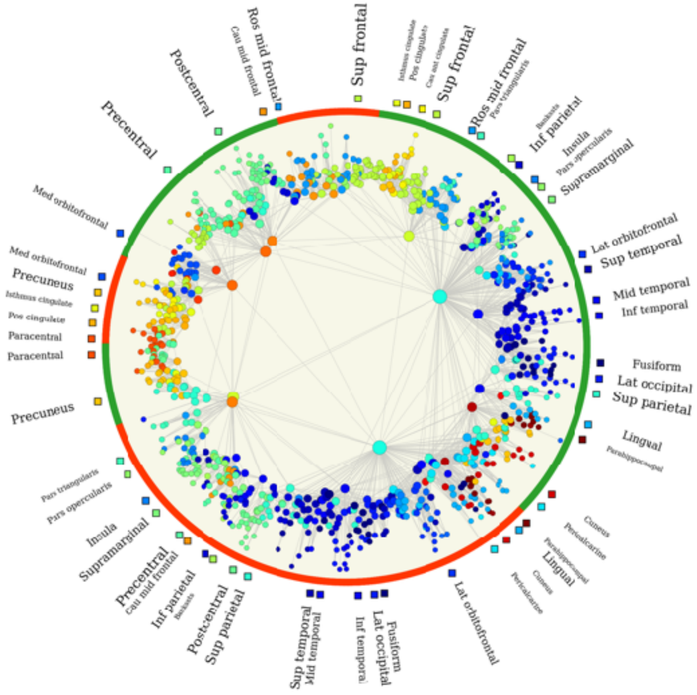
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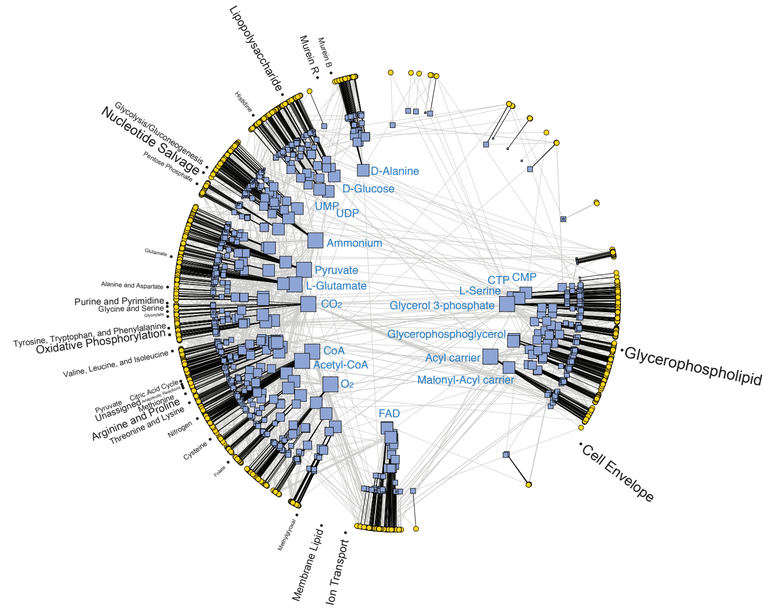
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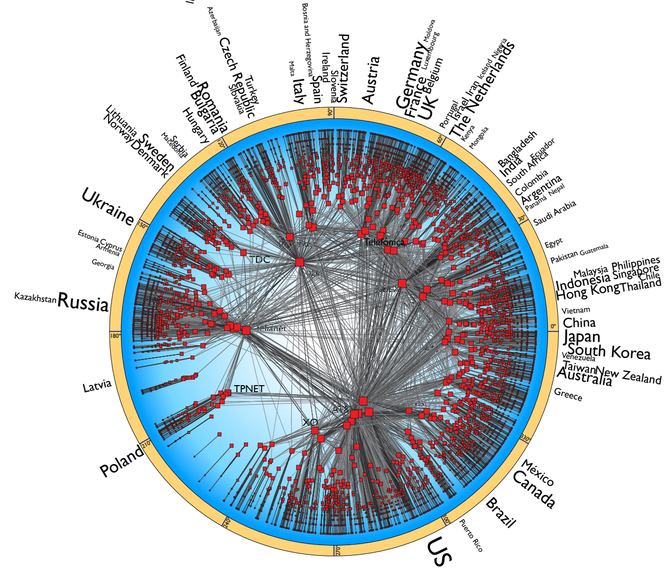
A powerful and versatile framework

- Amenable to many analytical calculations [1,2,3]
- Generalizable to weighted [5], bipartite [6,7,8], multiplex [9,10] and growing [11] networks
- Geometrical interpretation of preferential attachment [11]
- Parsimonious explanation of self-similarity [4]
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  - Reproduction of additional properties than the ones used to fit the parameters [15].
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Review Article | Published: 29 January 2021

Network geometry

Marián Boguñá, Ivan Bonamassa, Manlio De Domenico ✉, Shlomo Havlin, Dmitri Krioukov & M. Ángeles Serrano

Nature Reviews Physics **3**, 114–135 (2021)

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Three challenges in modeling directed networks

Properties of any metric space

Identity of indiscernibles	$d(x,y) = 0 \quad \Leftrightarrow \quad x = y$
Non-negativity	$d(x,y) \geq 0$
Symmetry	$d(x,y) = d(y,x)$
Triangle inequality	$d(x,y) \leq d(x,z) + d(z,y)$

with community structure[29,32,115]. The model has yet to be extended to directed networks, because it is not clear how to reconcile the intrinsic symmetry of metric distances with asymmetric interactions among nodes.

Nat. Rev. Phys. 3, 114 (2021)

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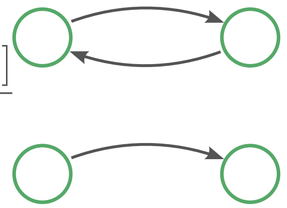
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Nat. Rev. Phys. 3, 114 (2021)

Reciprocity : cycles of length 2

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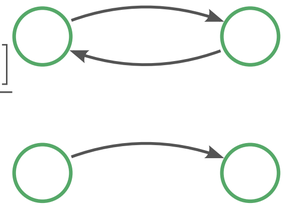
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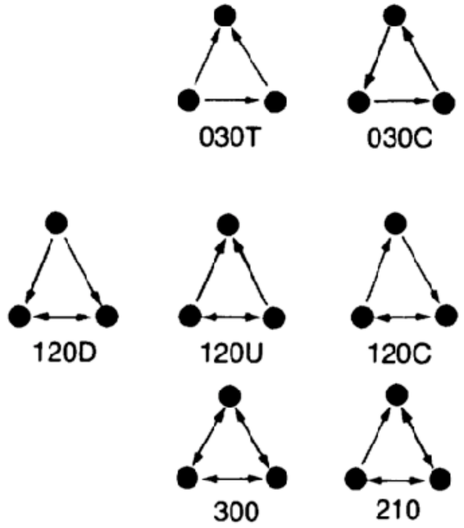
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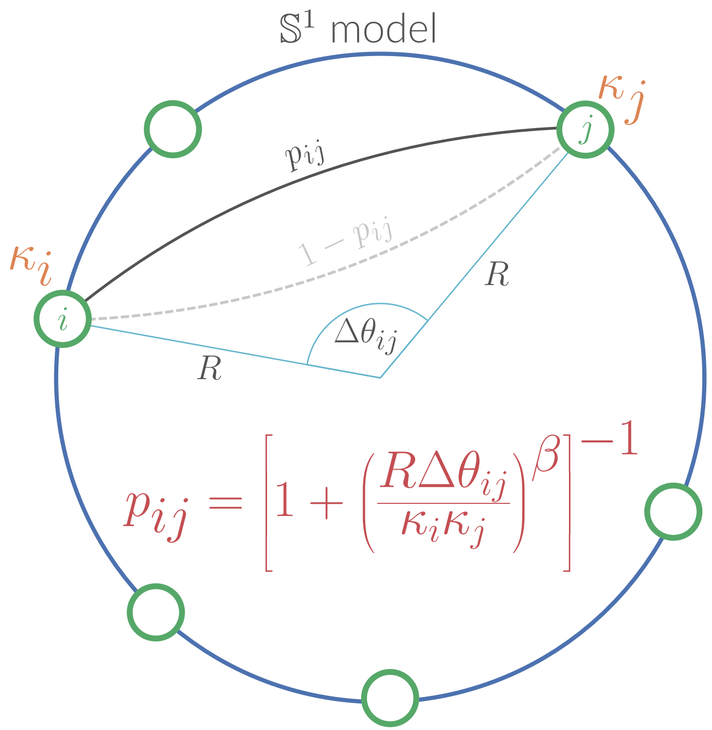


Clustering : 7 cycles of length 3

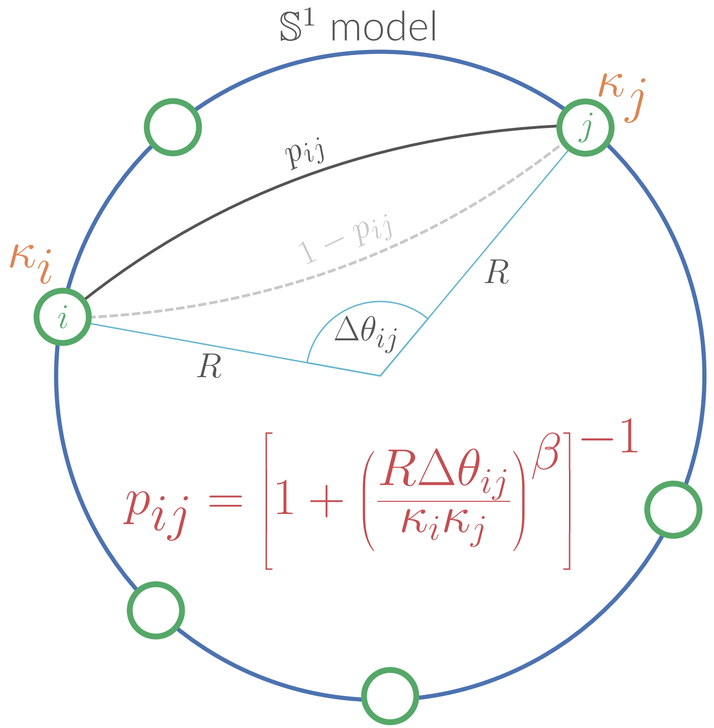


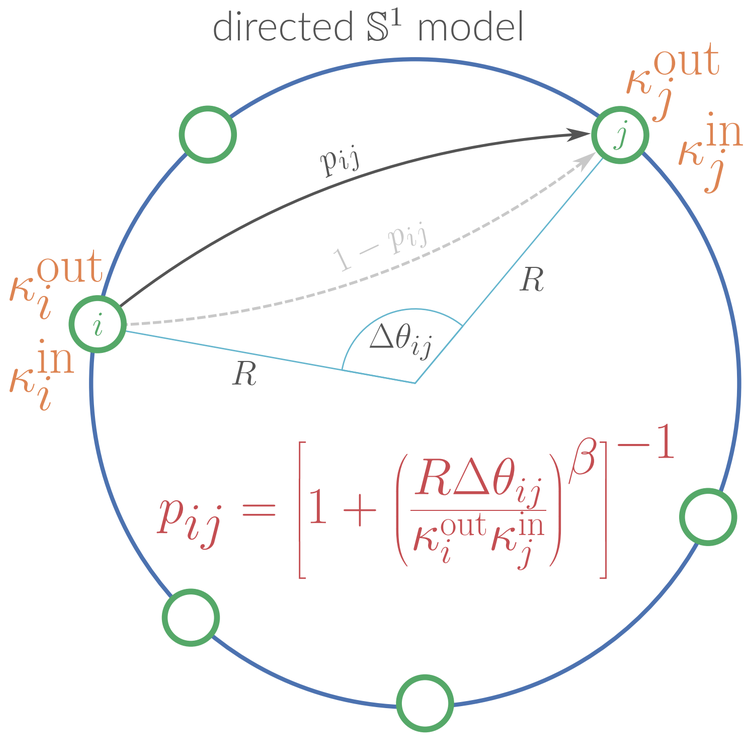
Adapted from Holland & Leinhardt. Local Structure in Social Networks. Sociol. Methodol., 7, 1–45 (1976)

The directed $\mathbb{S}^1$ model: A straightforward generalization



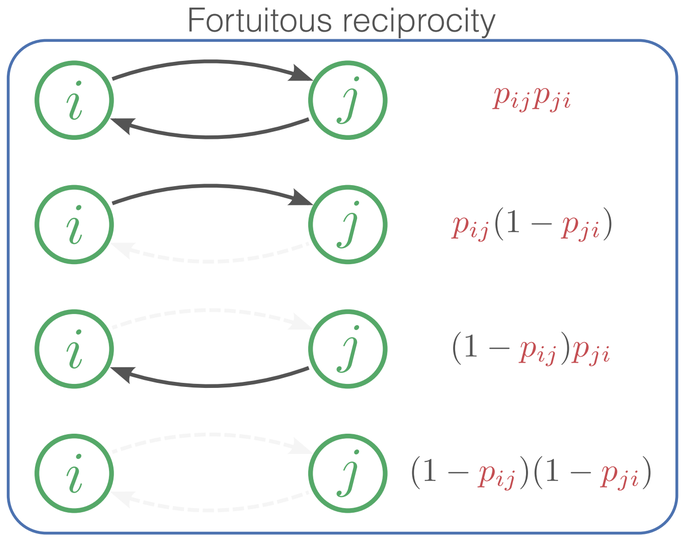
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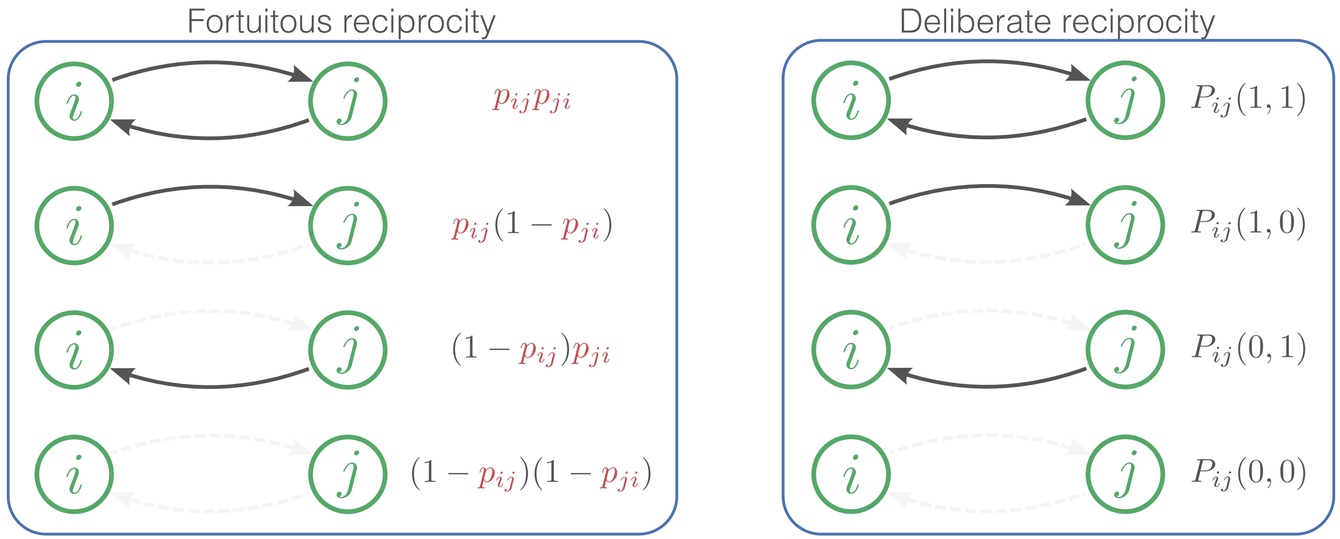
The directed-reciprocal $\mathbb{S}^1$ model: A new link between connections and distances

A random network model defines the probability p_{ij} for a directed link to exist from node i to node j.



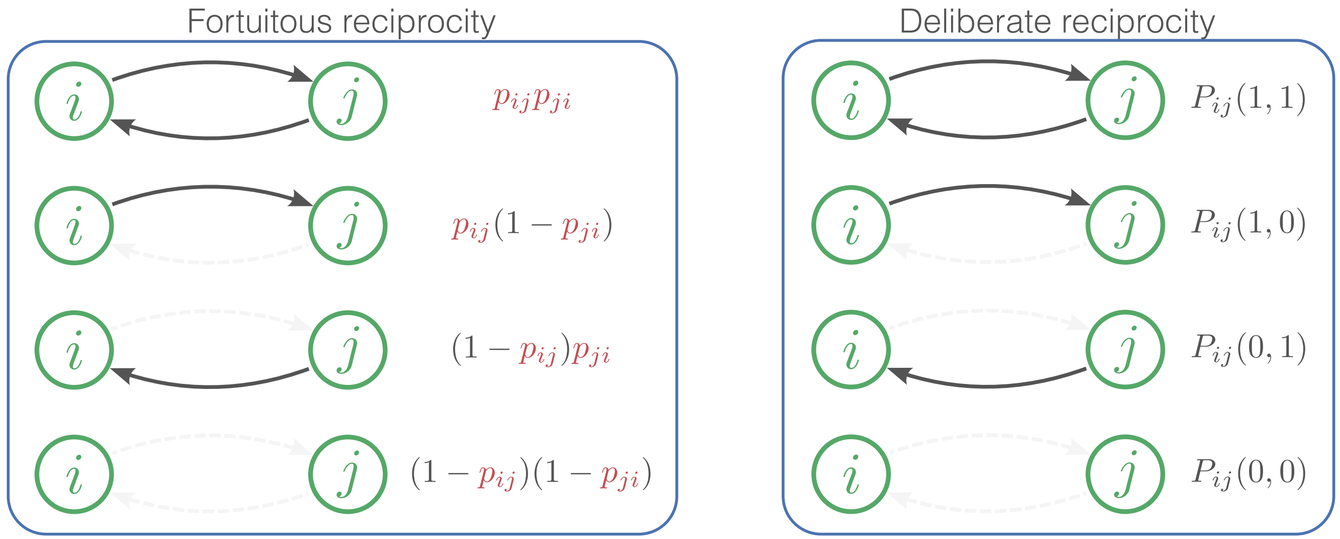
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Level of reciprocity controlled with parameter $-1 \leq \nu \leq 1$

$$P_{ij}(1,1) = \begin{cases} (1-\nu)p_{ij}p_{ji} + \nu \min\{p_{ij}, p_{ji}\} & 0 \leq \nu \leq 1 \\ (1+\nu)p_{ij}p_{ji} + \nu(1-p_{ij}-p_{ji})H(p_{ij}+p_{ji}-1) & -1 \leq \nu \leq 0 \end{cases}$$

$\nu = 1$: maximal reciprocity

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Condition 1: Preserves marginal probabilities

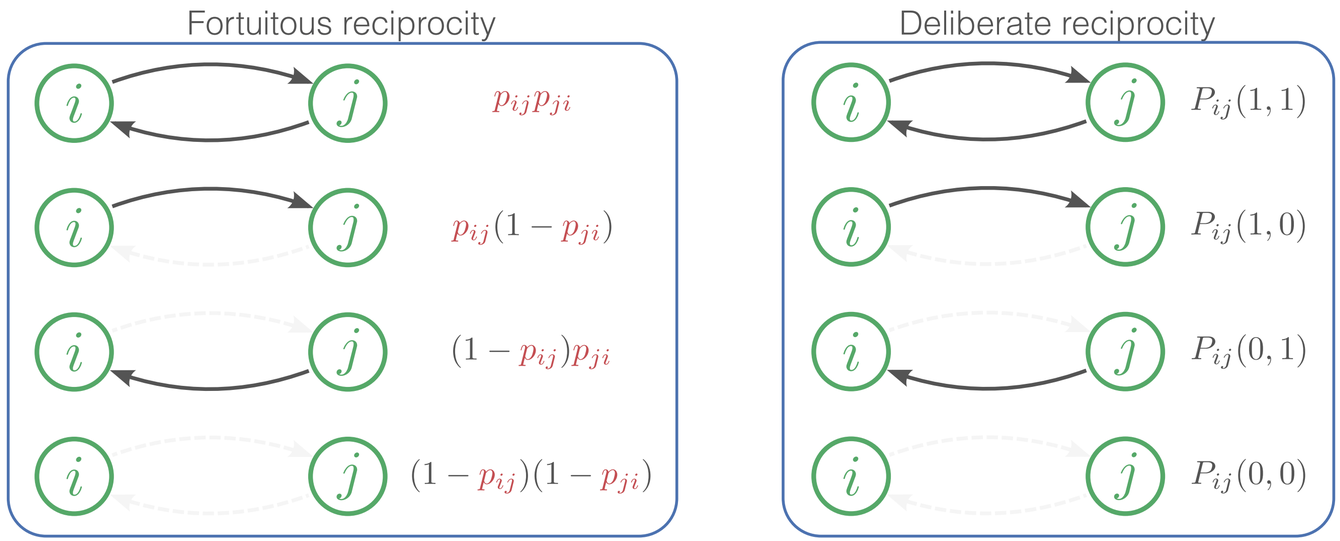
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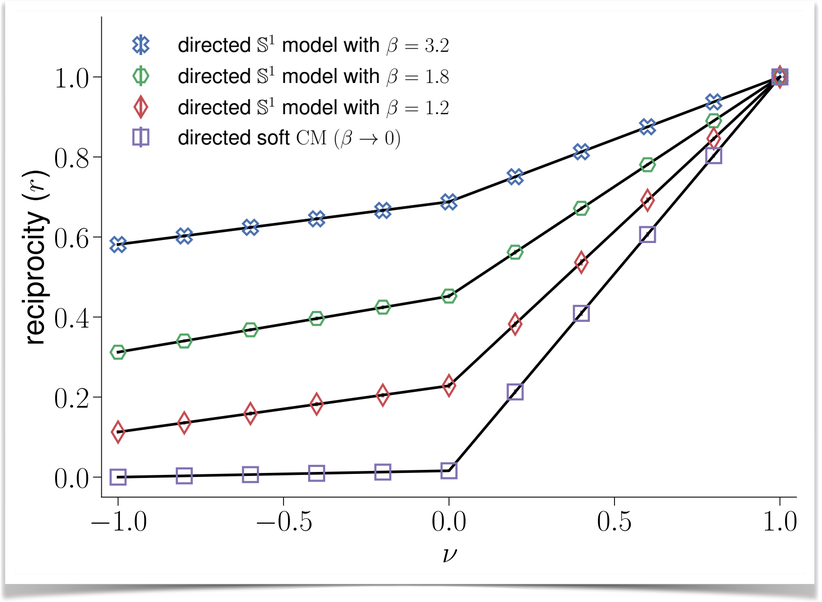
Condition 2: Normalized

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Fitting the directed-reciprocal $\mathbb{S}^1$ model to real networks

The likelihood of a network in the directed-reciprocal S^1 model is

$$\mathbb{P}\Big(\overbrace{\mathbb{A}}^{\text{adj. mat.}} \;\Big|\; \underbrace{\{\kappa_i^{\text{in}}\}, \{\kappa_i^{\text{out}}\}, \{\theta_i\}}_{\text{coordinates}}, \overbrace{\beta, \nu}^{\text{param.}} \Big) = \underbrace{\prod_{i=1}^{N} \prod_{j=i+1}^{N} P_{ij}(a_{ij}, a_{ji})}_{\text{likelihood}}$$

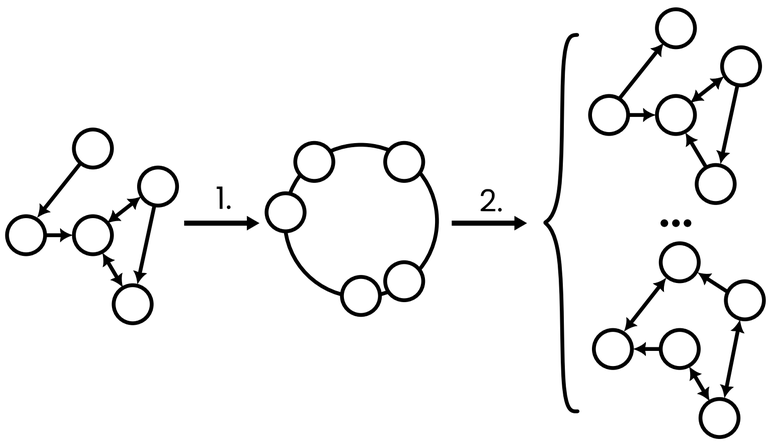
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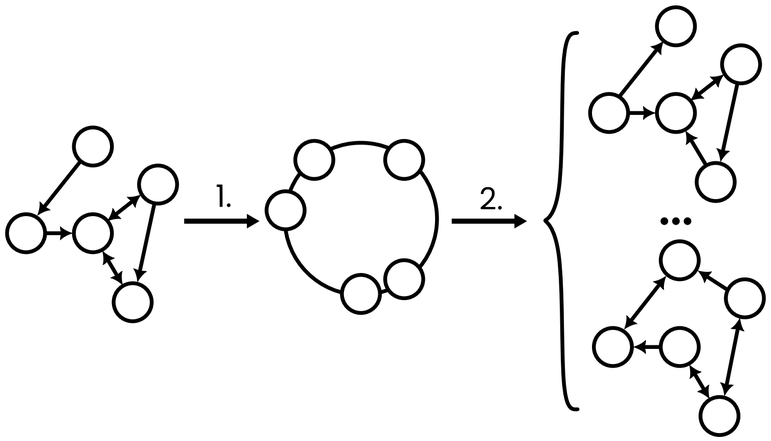


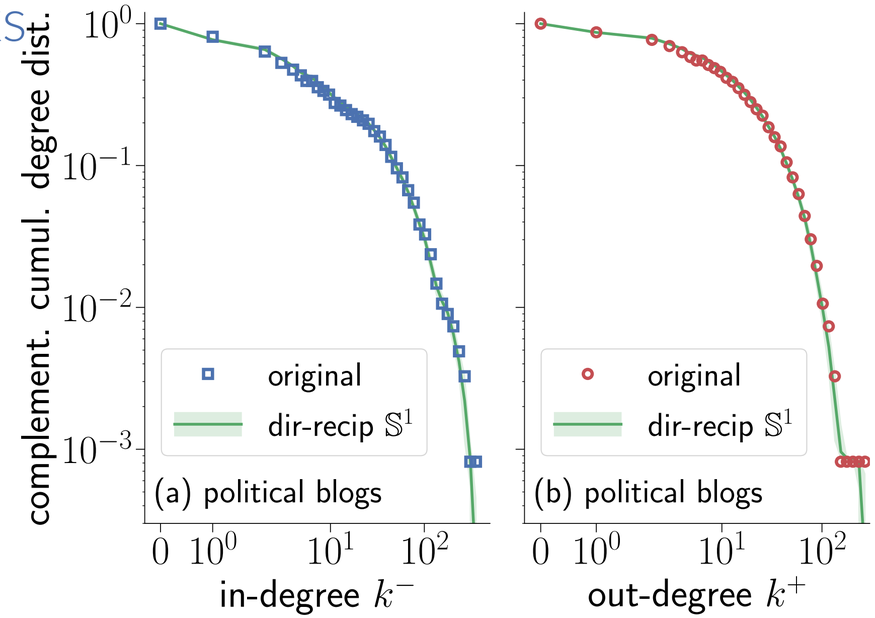
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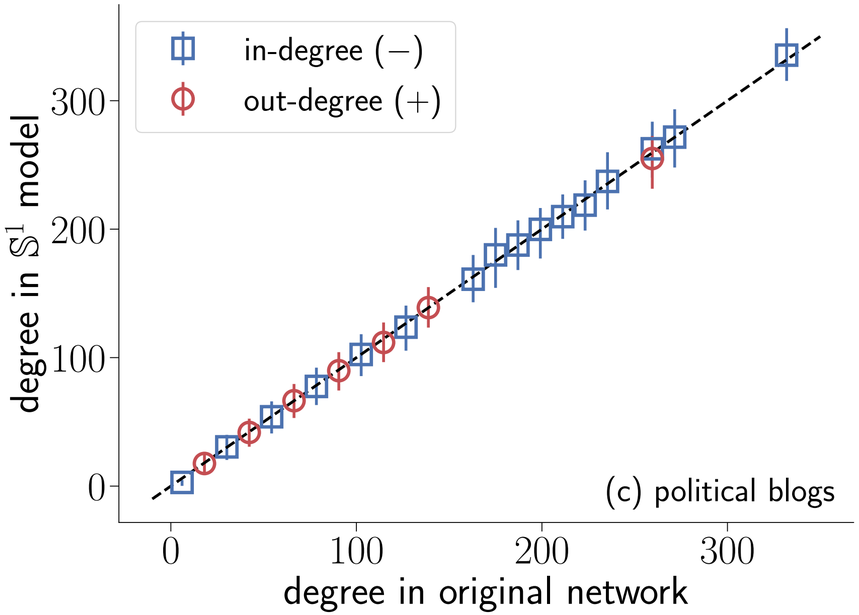
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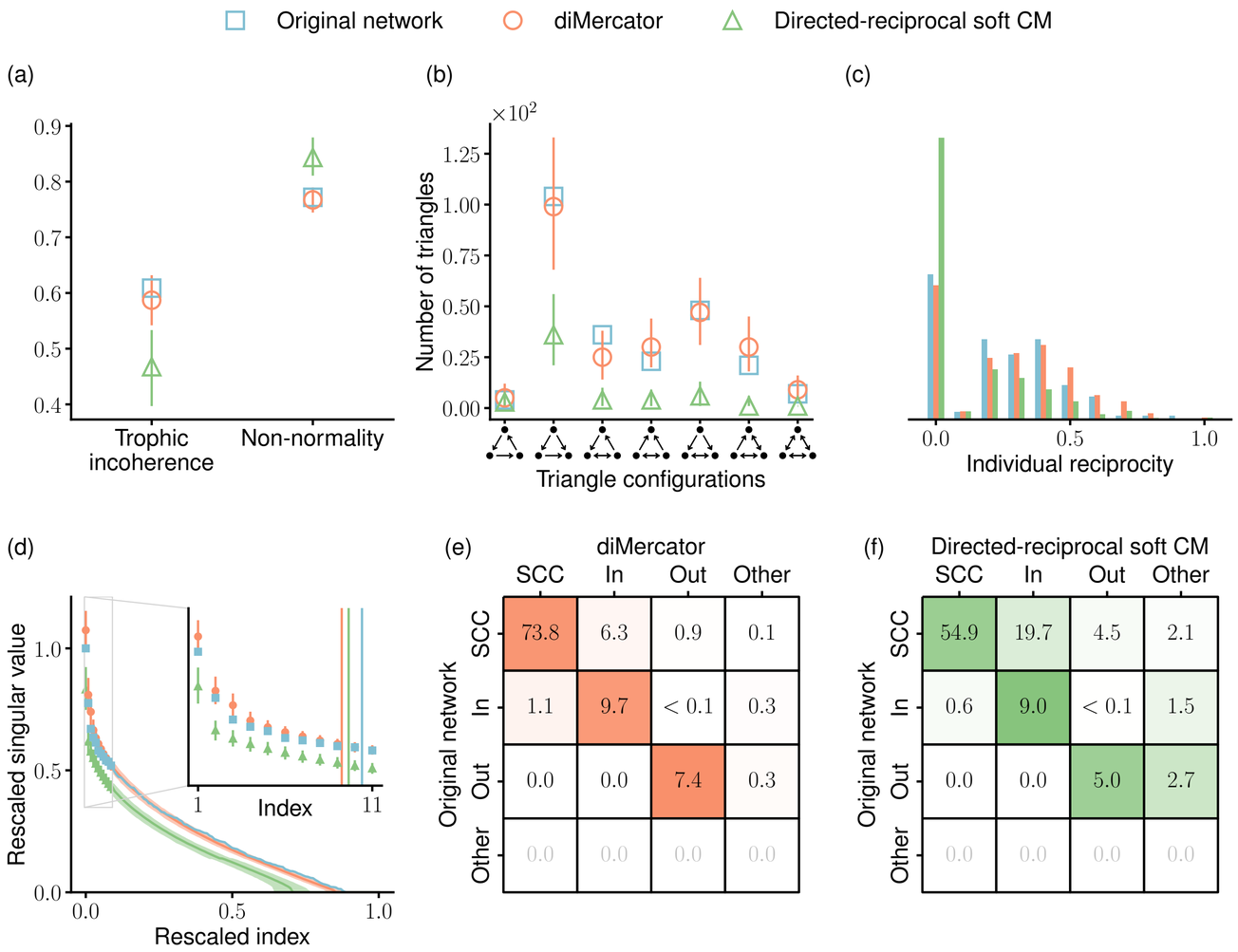
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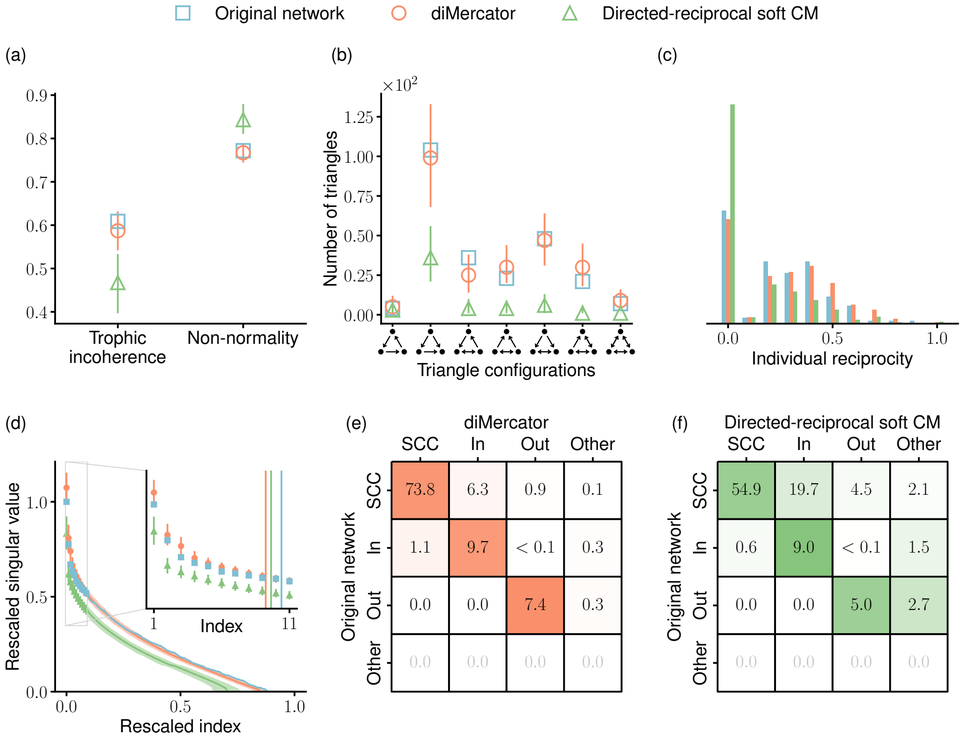




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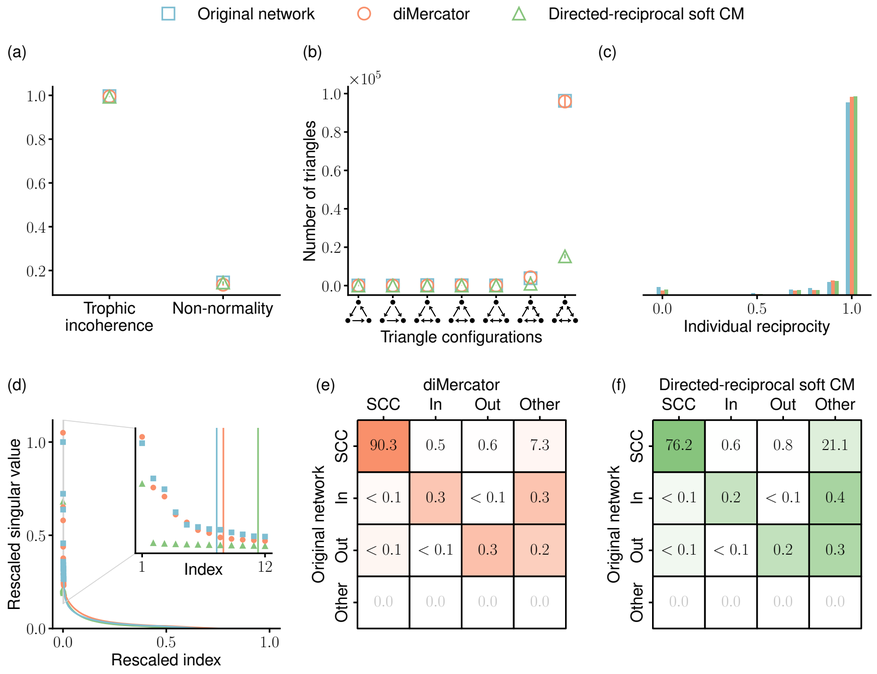


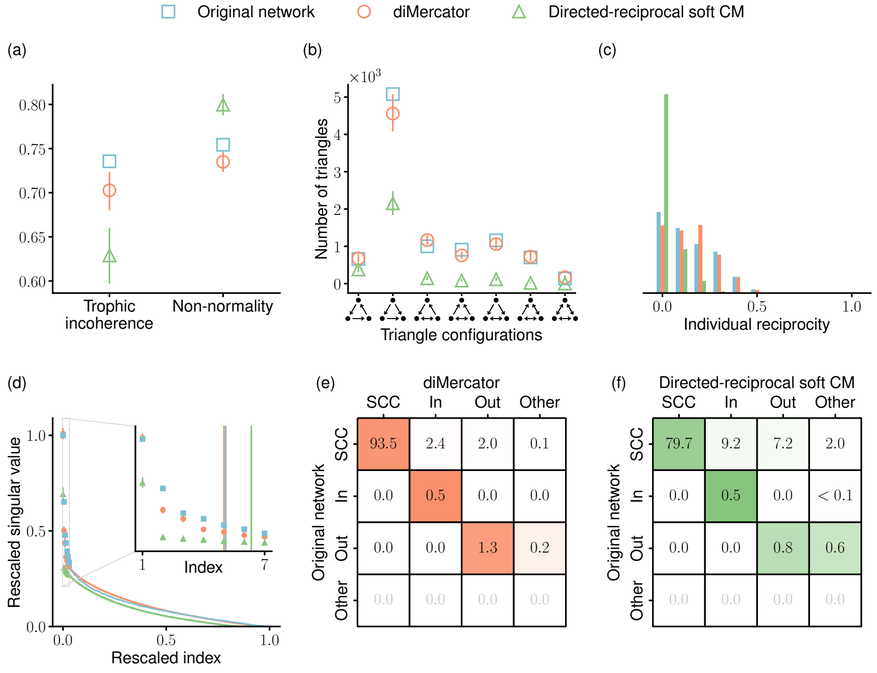
Fitting the directed-reciprocal $\mathbb{S}^1$ model to real networks



Fixing the degree sequence (on average), the global density of triangles and global reciprocity strongly constrains

- the non-normality [1]
- the trophic coherence [2]
- the clustering patterns
- the individual reciprocities
- the singular values
- the bow-tie organization





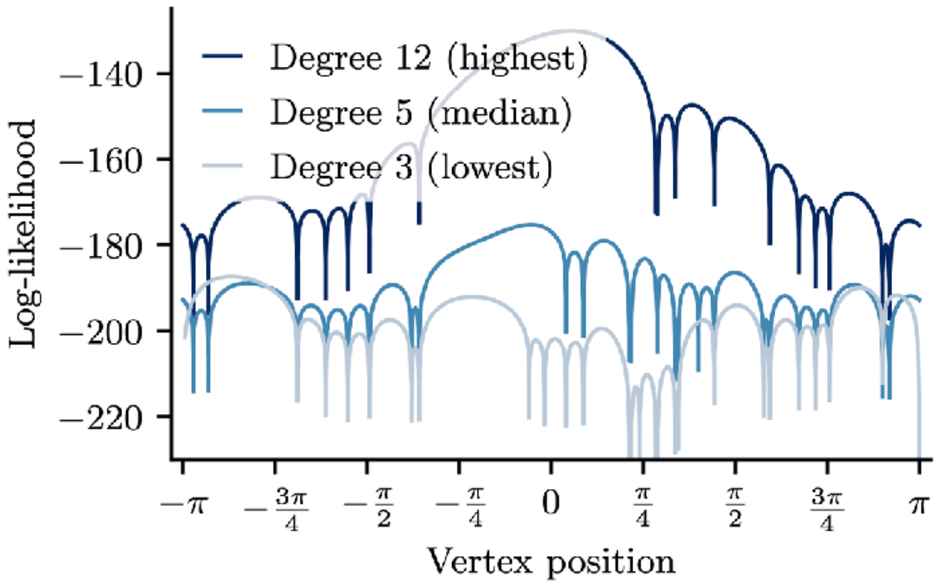
[1] Sci. Adv. 4, eaau9403 (2018)
[2] PNAS 111, 17923 (2014)

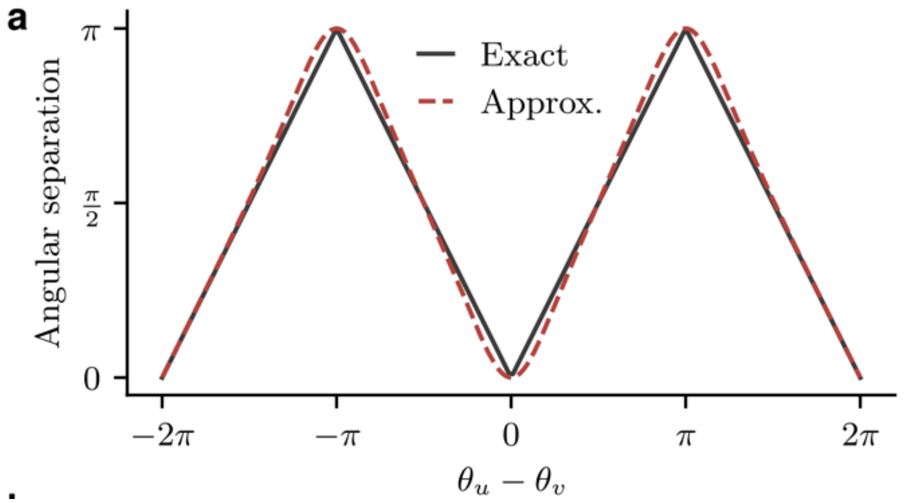
Challenges

Heterogeneous random geometric graph models are prime candidates to model real networked complex systems.

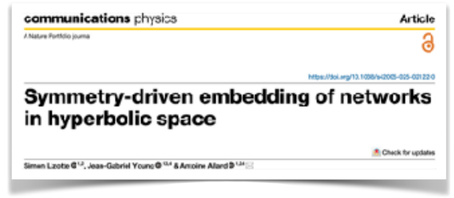
But they rely heavily on our capacity to find high-quality embeddings of the original datasets.

- ▷ Difficult optimization problem
  - rugged landscape
  - numerous symmetries (rotation, reflection, graph automorphisms)
  - gradient not always well defined
- ▷ Out-of-the-box solutions do not work well
  - Hamiltonian Monte Carlo
  - gradient descent
- ▷ Current state-of-the-art embedding methods
  - rely on heuristics
  - do not provide uncertainties (loglikelihood maximization)

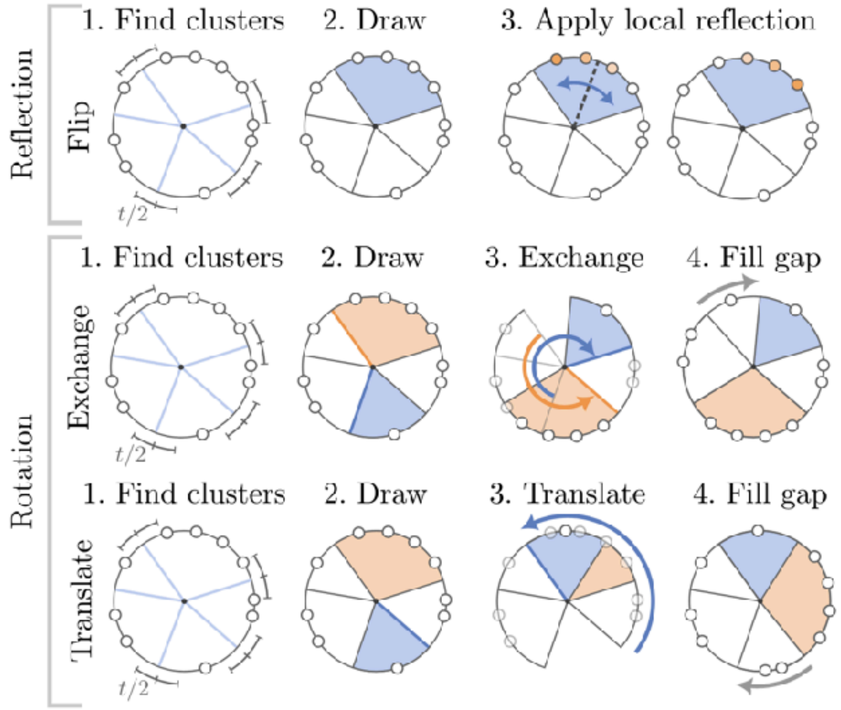




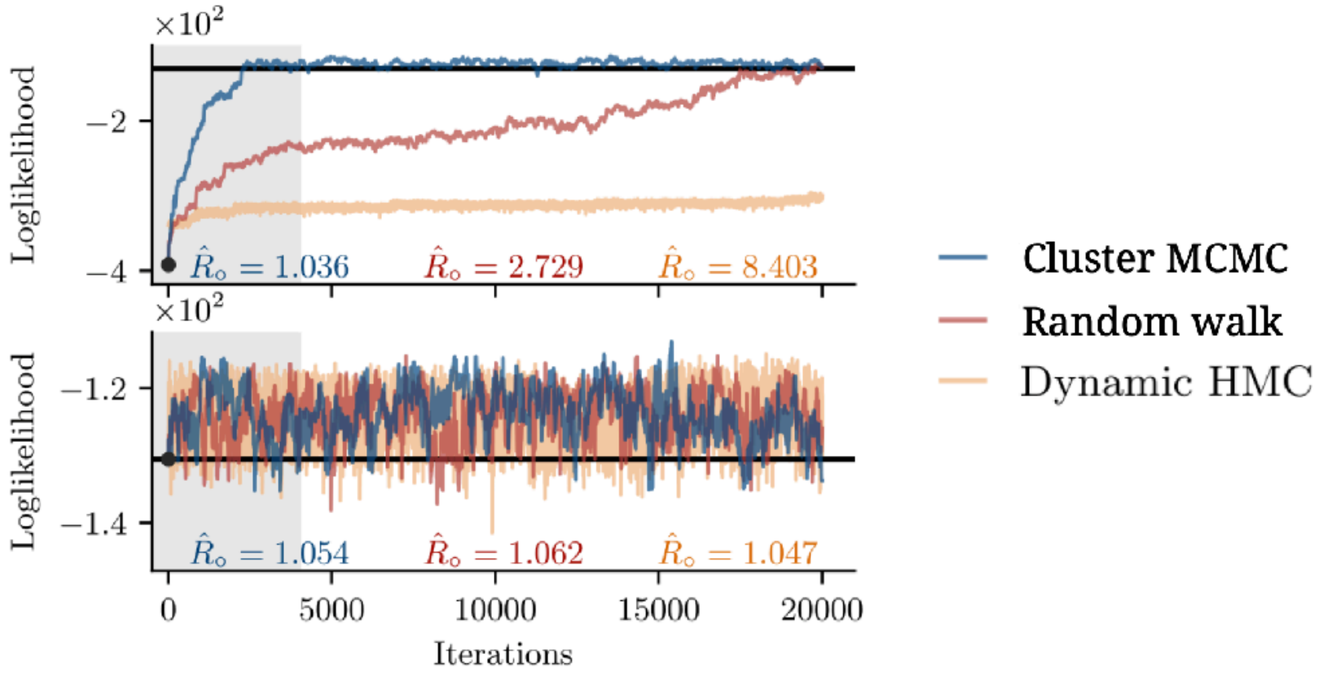
Challenges: some solutions



Leverage the specificities of the model to design better sampling algorithm.



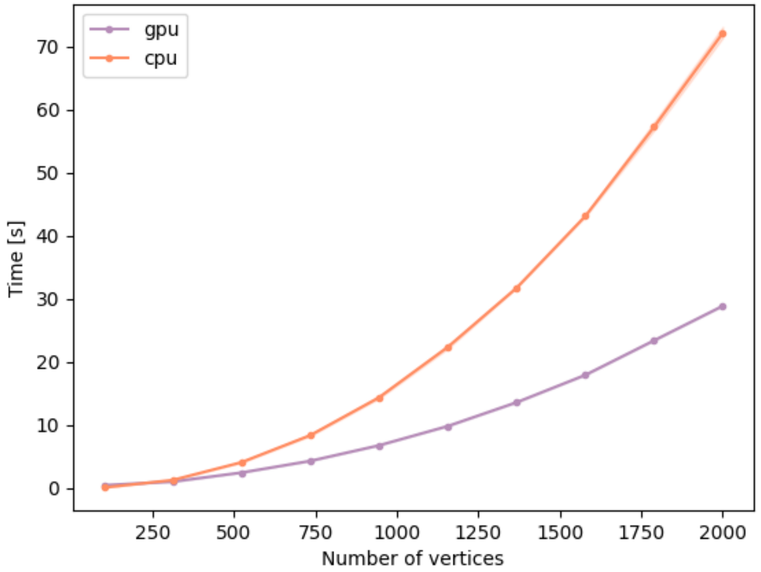
Better mixing and exploration than standard sampling algorithms.



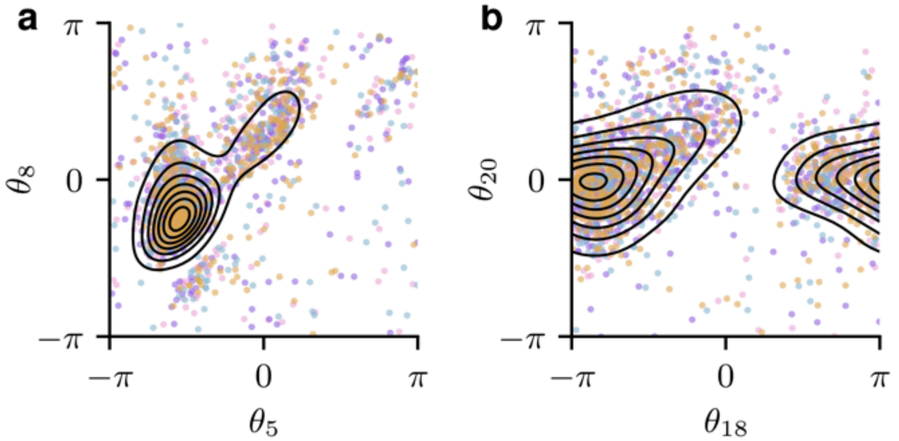
Challenges: some solutions

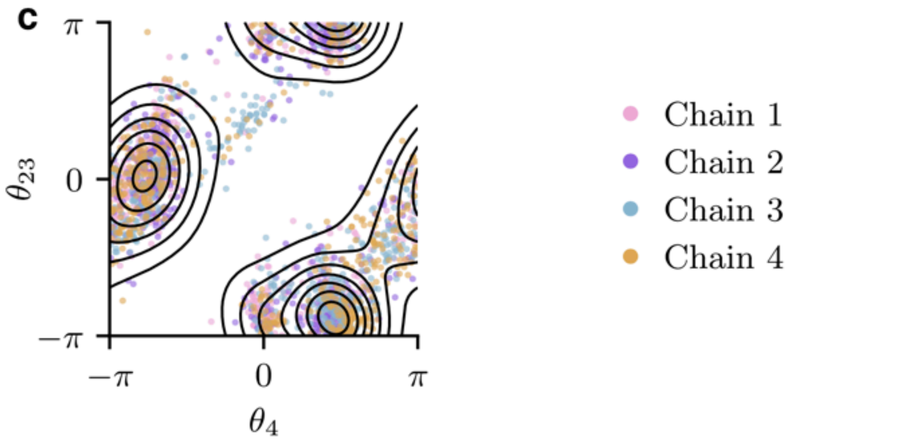


Still slow…

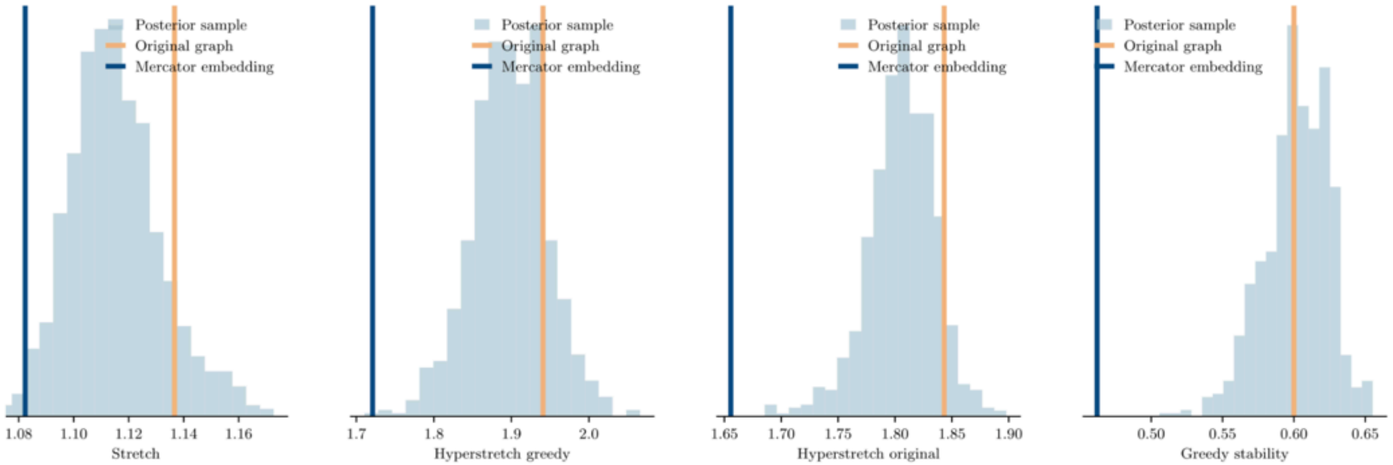


…and these uncertainties are not "trivial" (non-normal).



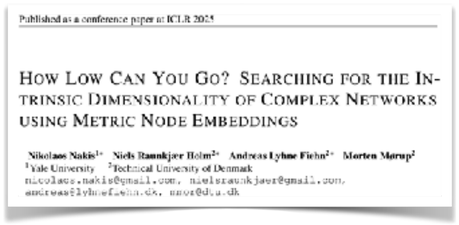


…but we now have uncertainties on graph metrics!



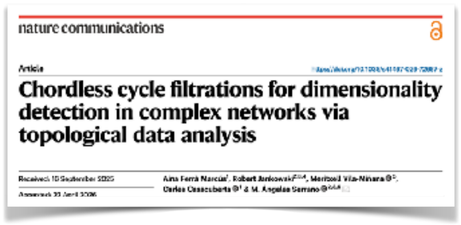
Challenges

What dimension should the embedding space have?


Published as a conference paper at ICLR 2025

HOW LOW CAN YOU GO? SEARCHING FOR THE INTRINSIC DIMENSIONALITY OF COMPLEX NETWORKS USING METRIC NODE EMBEDDINGS

Nikolaos Nakis[1*] Niels Raunkjær Holm[2*] Andreas Lyhne Fiehn[2*] Morten Mørup[2]
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nikolaos.nakis@gmail.com, nielsraunkjaer@gmail.com,
andreasfiehn@hotmail.dk, mmor@dtu.dk


nature communications

Article

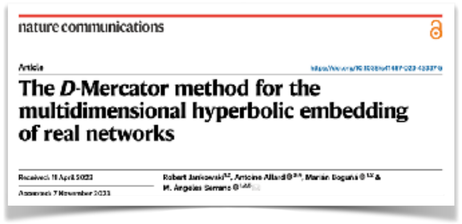
Chordless cycle filtrations for dimensionality detection in complex networks via topological data analysis


PNAS nexus

PNAS Nexus, 2023, 2, 1–12
https://doi.org/10.1093/pnasnexus/pgad136
Advance access publication 18 April 2023
Research Report

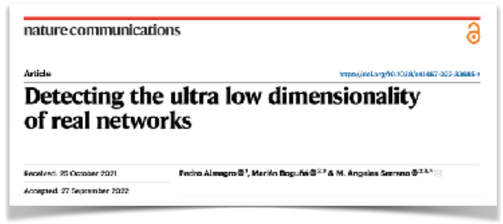
Dimension matters when modeling network communities in hyperbolic spaces

Béatrice Déay, Patrick Desrosiers and Antoine Allard


nature communications

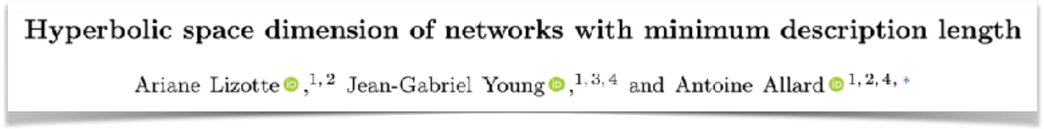
Article

The *D*-Mercator method for the multidimensional hyperbolic embedding of real networks

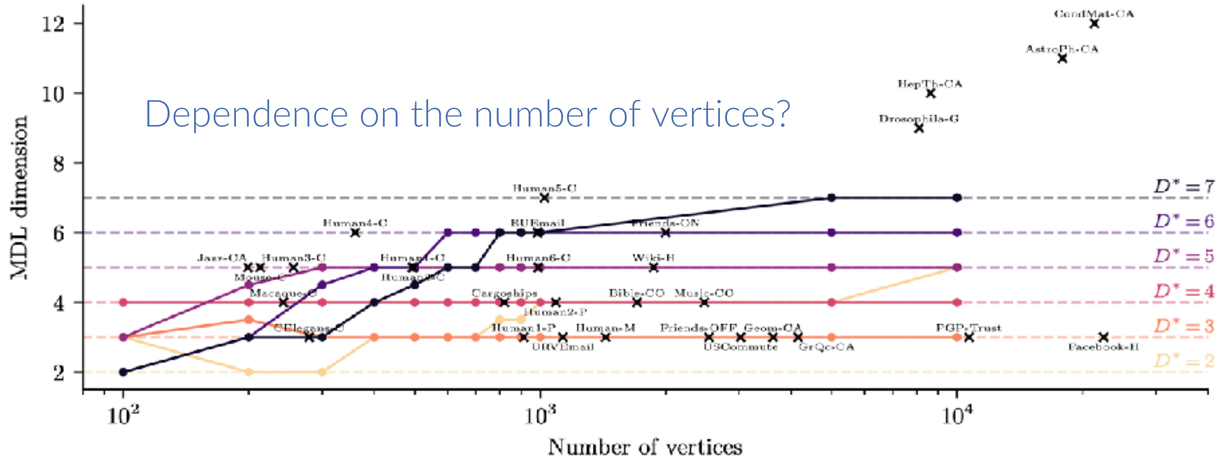

nature communications

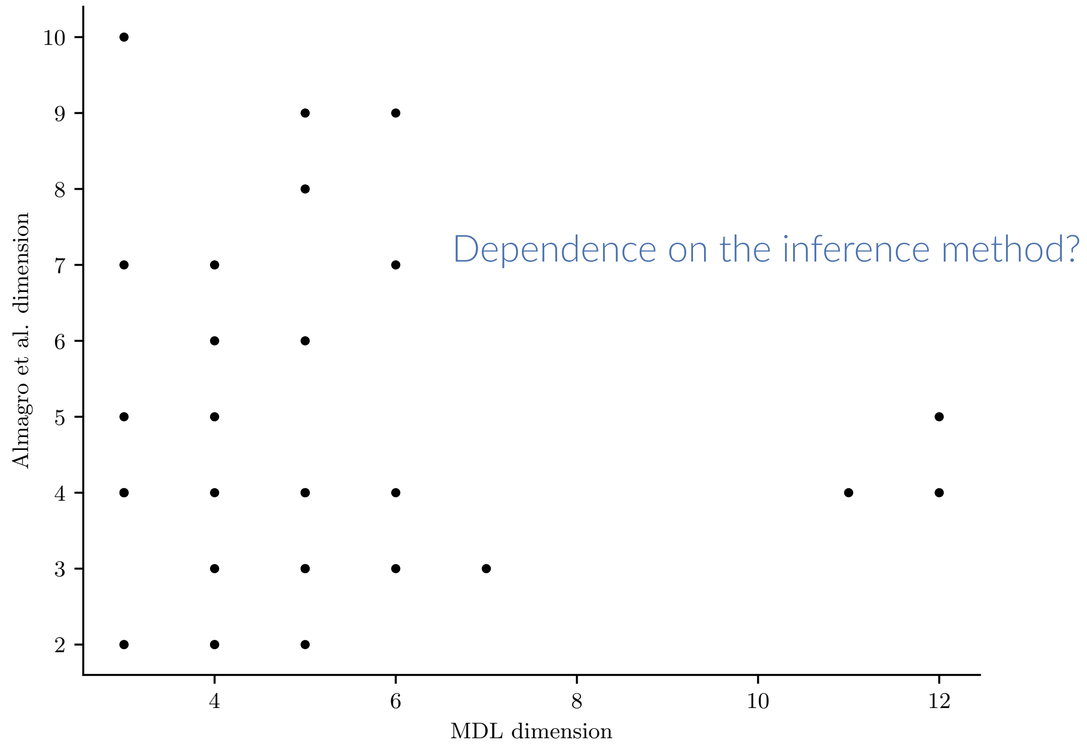
Article

Detecting the ultra low dimensionality of real networks


Hyperbolic space dimension of networks with minimum description length

Ariane Lizotte[1,2] Jean-Gabriel Young[1,3,4] and Antoine Allard[1,2,4,*]





Main takeaways

- Rethinking the link between connections and distances allows the extension of the network geometry framework to directed networks.
- Assuming an underlying geometry is a simple modeling approach that can reproduce several complex topological properties observed in real networks.

Challenges

- Network geometry is a promising framework, but its applicability relies heavily on our capacity to find high-quality embeddings of the original datasets, which is hard.
- We cannot power our way through this challenge using GPUs or supercomputers: we need to design smarter algorithms.



M. Ángeles Serrano
Universitat de Barcelona & ICREA



Jérémi Lesage
Université Laval

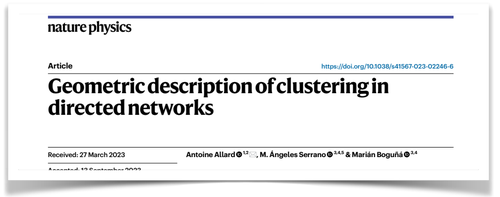
Marián Boguñá
Universitat de Barcelona



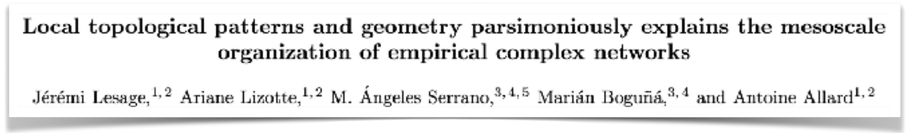


Jean-Gabriel Young
University of Vermont

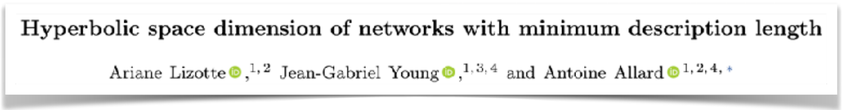
Ariane Lizotte
Université Laval







Preprints on arXiv expected fall 2026



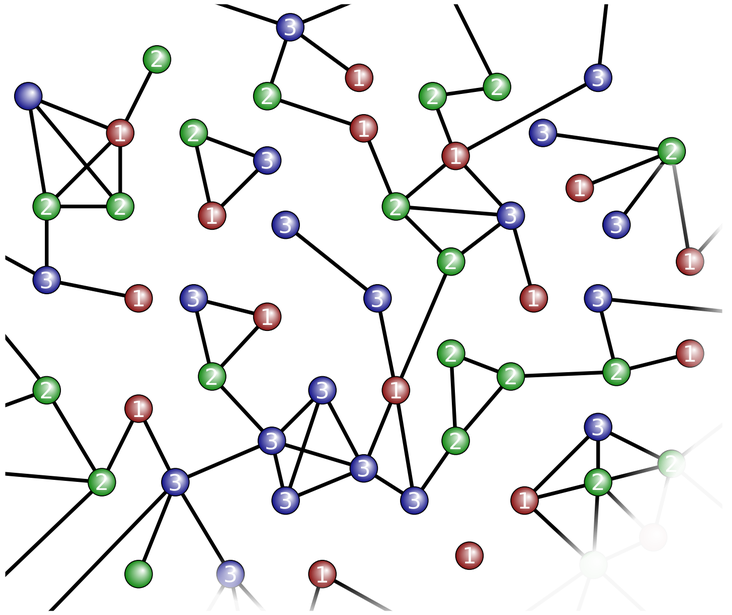
Outline

1. Are simple models enough to study complex systems/networks?

2. **“Simple” ways to encode structural complexity**

   (a) latent metric space

   (b) stub types



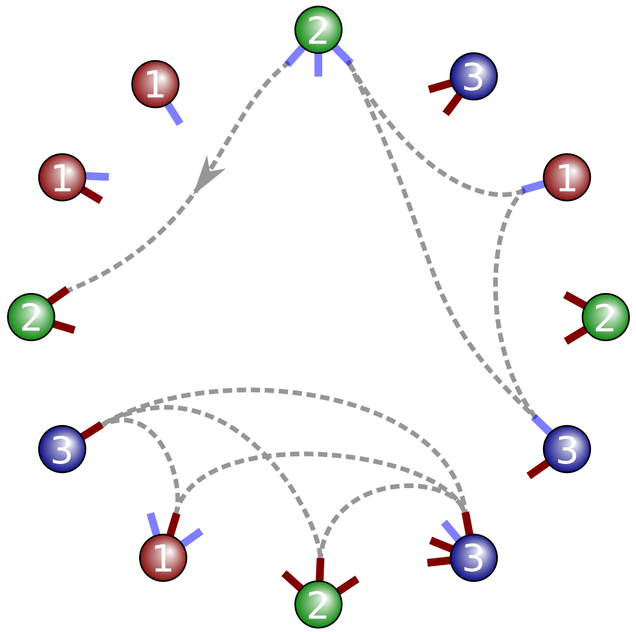
General *stub matching* scheme

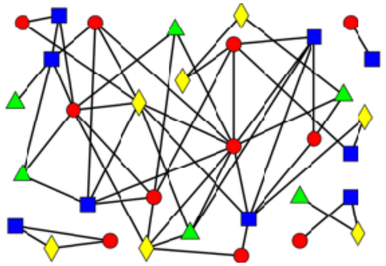
Many network models can be seen as a *stub matching* scheme with

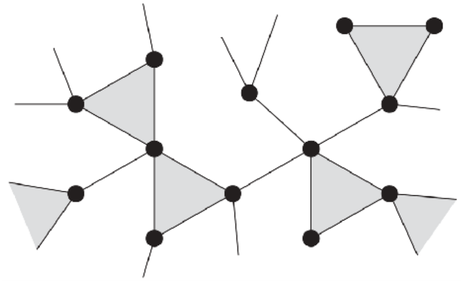
- node types
- stub types
- rules governing how stubs are matched

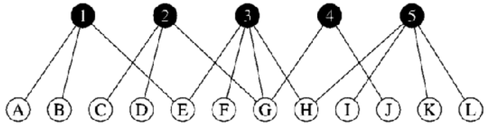
This perspective facilitates the mathematical description of the dynamical processes on networks

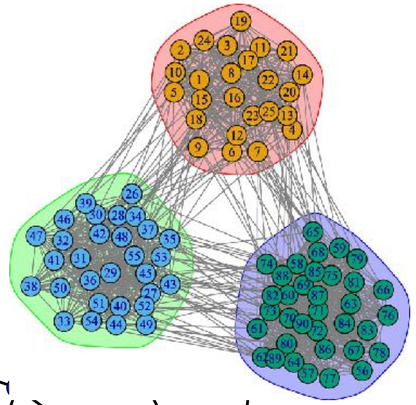
- probability generating functions (ex.: percolation, robustness)
- ordinary differential equations (ex.: epidemic spreading, opinion dynamics)

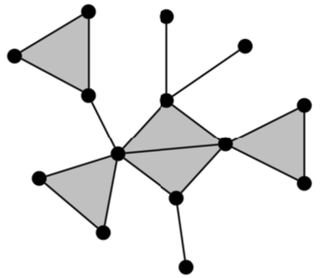


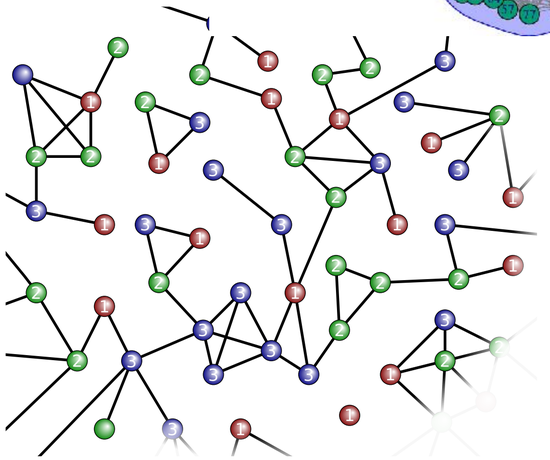












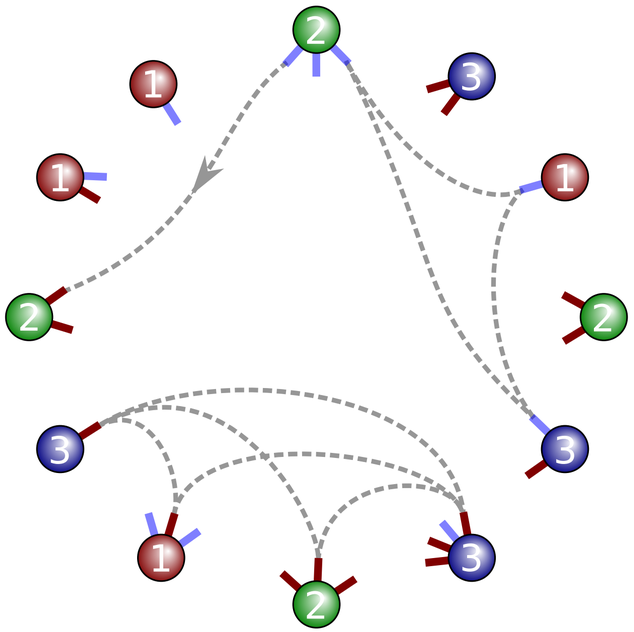
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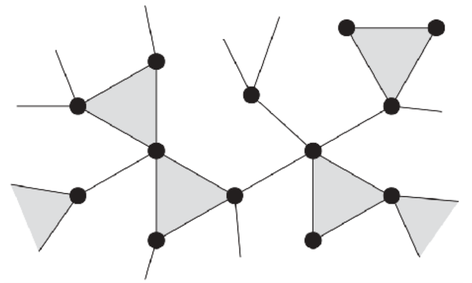
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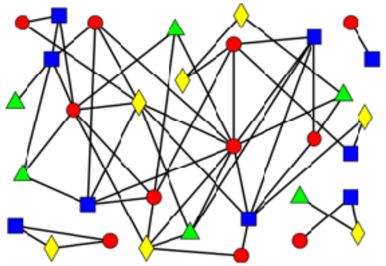
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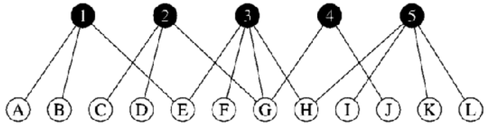
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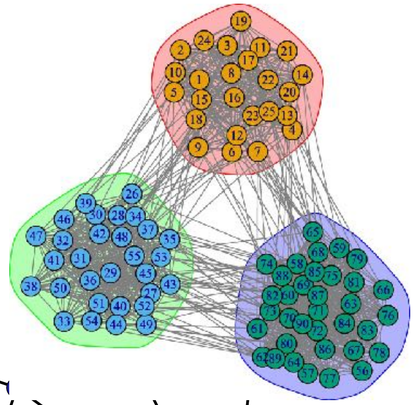
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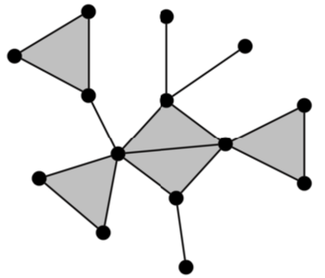


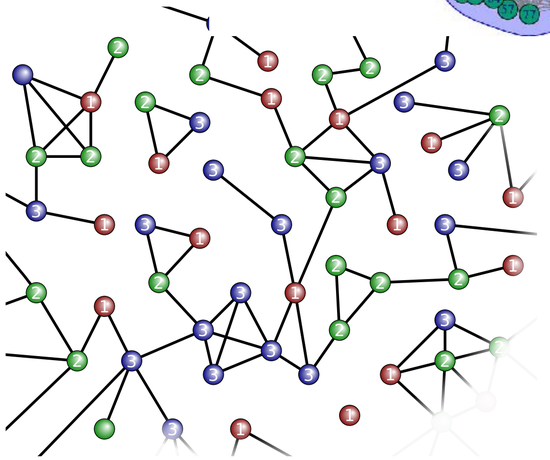










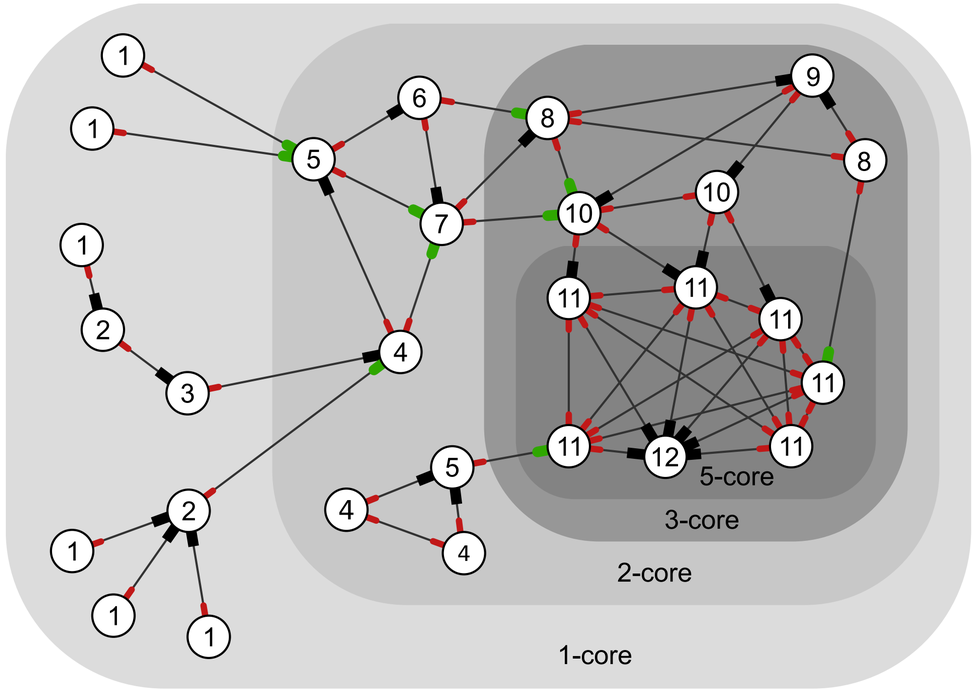


Most of these approaches use stub types to enforce *local* connection patterns.

Mesoscopic level: The k-core/onion decomposition

Onion decomposition: k-core decomposition with additional information about the positions of nodes within every k-shell (layers).

Information about layers is obtained from the k-core decomposition with minimal additional computational cost.



Mesoscopic level: The k-core/onion decomposition

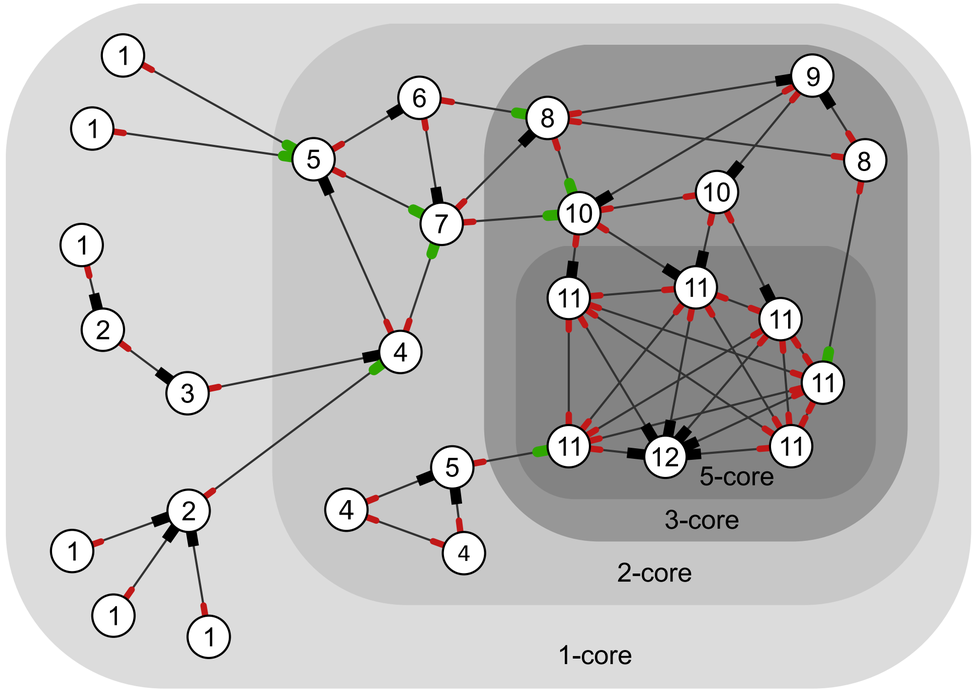
Onion decomposition: k-core decomposition with additional information about the positions of nodes within every k-shell (layers).

Information about layers is obtained from the k-core decomposition with minimal additional computational cost.

Stub matching scheme

- ▷ One type of nodes per layer
- ▷ Three types of stubs (red, green, black)
- ▷ Rules:

1. Allowed links: red–red, red–green, red–black
2. Nodes in layer ℓ and shell k must
   (a) have exactly k links to nodes in layers $\ell' \geq \ell$ (if layer ℓ is the first layer of the k-shell).
   (b) have at least $k+1$ links to nodes in layers $\ell' \geq \ell - 1$ and at most k links to nodes in layers $\ell' \geq \ell$ (if it is not in the first layer of the k-shell).



Mesoscopic level: The k-core/onion decomposition

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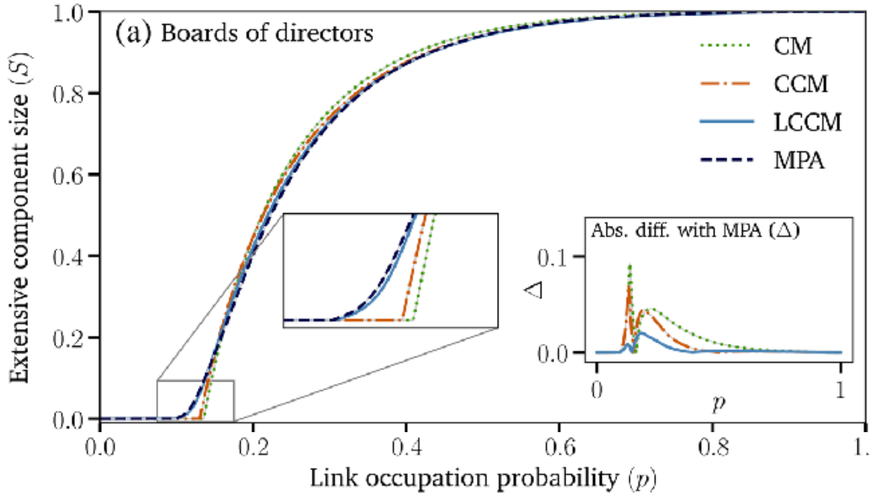
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The onion decomposition accounts for a lot when it comes to the precision of message-passing description of percolation.



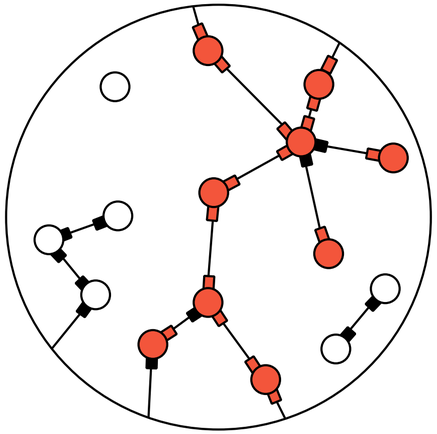
Macroscopic level: Connectivity

Preprocessing: Identify links that lead to the largest connected component and tag them as red; all remaining stubs are tags as black.

Stub matching scheme

- One type of nodes
- Two types of stubs (red, black)
- Rules:

1. Allowed links: red-red, red–black
2. Black stubs connect to any other stubs, but only towards nodes with excess red degree 0.
3. Red stubs connect to any other stubs, but only towards nodes with excess red degree at least 1.



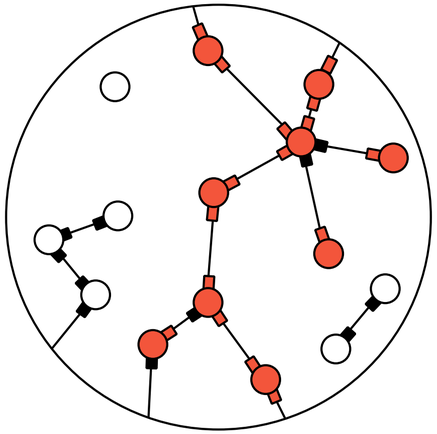
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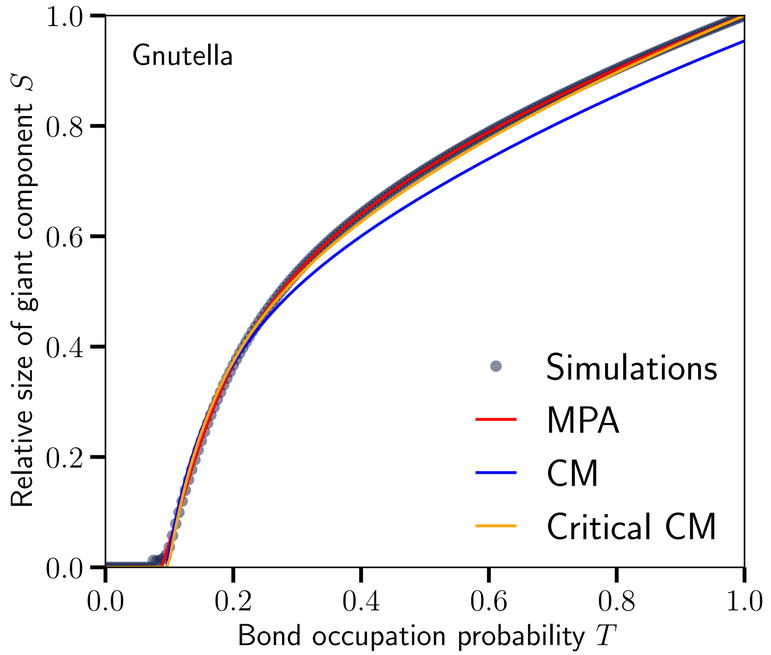
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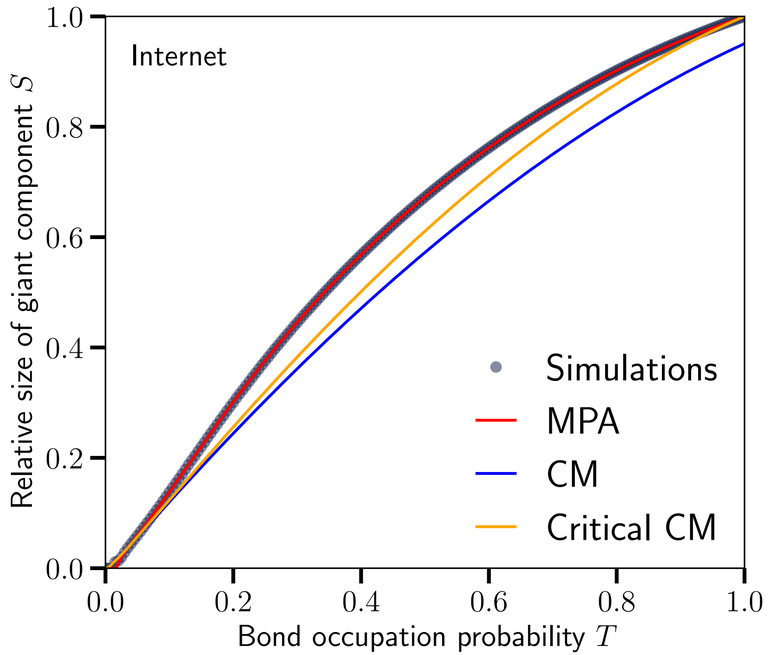
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Main takeaways

- Local connection rules can enforce strict global topological features.
- These rules can be leveraged to design
  - new mathematical frameworks (ex.: ODEs, PGFs)
  - new sampling algorithms (ex.: edge swapping)

Challenges

- Many analytical approaches (PGF, ODE) can be adapted to account for stub types, but the graphs considered are infinite in size.
- Sampling naively is easy, sampling right is tricky (ex.: uniform sampling via edge swaps).

Multi-scale structure and topological anomaly detection via a new network statistic: The onion decomposition

Laurent Hébert-Dufresne[1], Joshua A. Grochow[2] & Antoine Allard[2]

Sci. Rep. 6, 31708 (2016)

Percolation and the Effective Structure of Complex Networks

Antoine Allard[1,2] and Laurent Hébert-Dufresne[3,1]

Phys. Rev. X 9, 011023 (2019)

PHYSICAL REVIEW RESEARCH 7, L022050 (2025)

Letter

Random graphs with specific degree distribution and giant component size

Laurent Hébert-Dufresne[1,2,3] Márton Pósfai[4] and Antoine Allard[3,5,1]

On the Uniform Sampling of the Configuration Model with Centrality Constraints

François Thibault,[1,2] Laurent Hébert-Dufresne,[3,4] and Antoine Allard[1,5]

Preprint on arXiv expected fall 2026



Laurent Hébert-Dufresne
University of Vermont

François Thibault
Université Laval





Márton Pósfai
Central European University

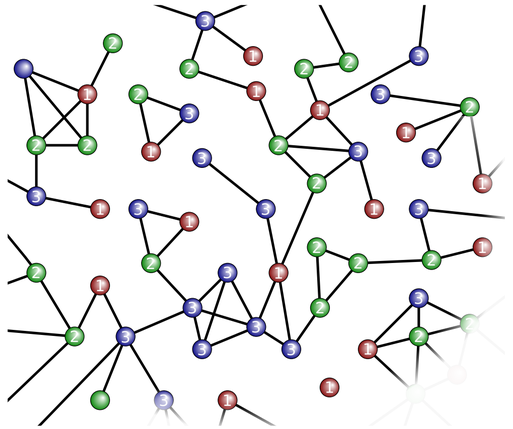
Joshua A. Grochow
University of Colorado, Boulder

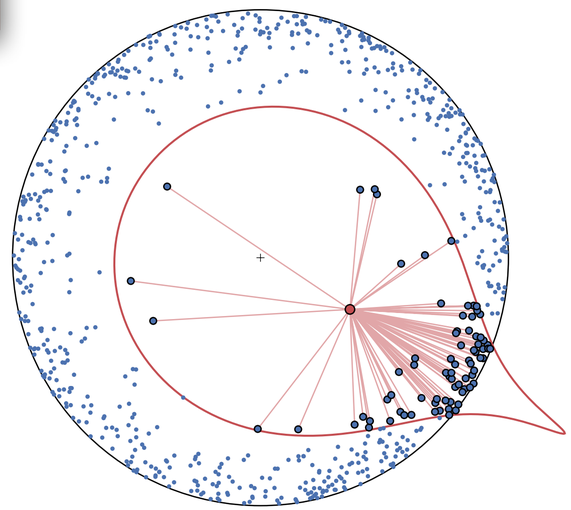


Theoretical justifications exist for the use of "simple" models of complex networks and of the dynamics they support.

Simple models can reproduce nontrivial complex connections patterns observed in real networks, but there are still several challenges ahead to bridge the gap between our models and real networks.







Dynamica

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Centre interdisciplinaire en modélisation mathématique de l'Université Laval

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Slides