Simplicity meets complexity

Reproducing connectivity patterns in complex systems with simple models

Antoine Allard

Université Laval, Québec, Canada

antoineallard.github.io

Power blackouts



2025 Iberian Peninsula blackout

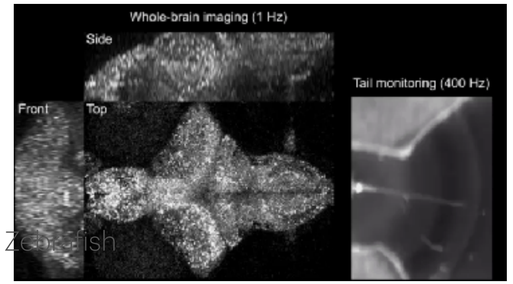


2003 Italy blackout

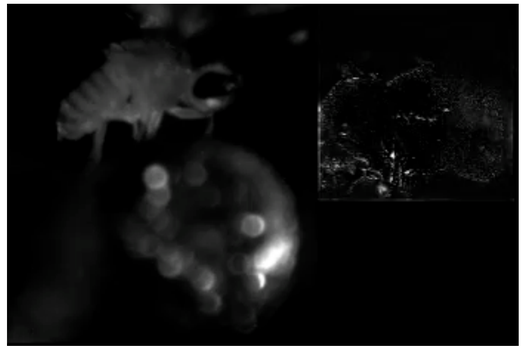


Northeast blackout of 2003

Cognition and motor control



Zebrafish



Drosophila

Traffic jams

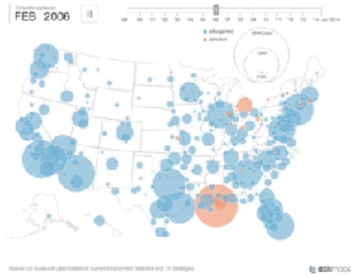


Economic crises



Wall Street crash of 1929

Great Recession



Animal collectives



Flock of birds

Epidemics



COVID-19 pandemic



2002–2004 SARS outbreak



Spanish flu



Western African Ebola epidemic



School of fish

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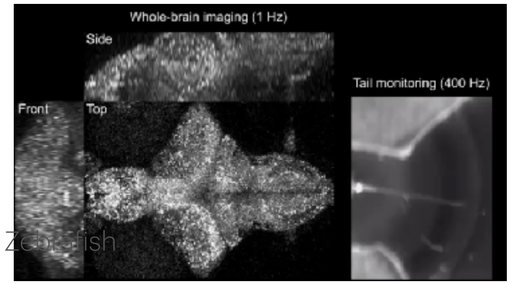


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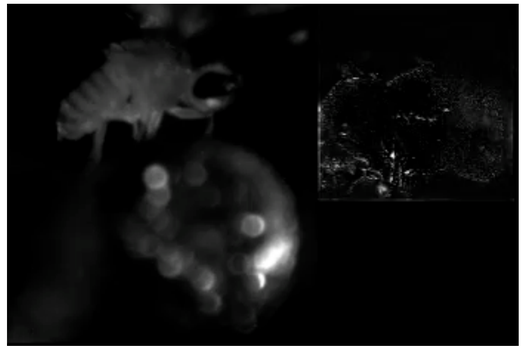


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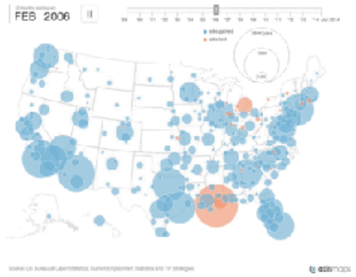


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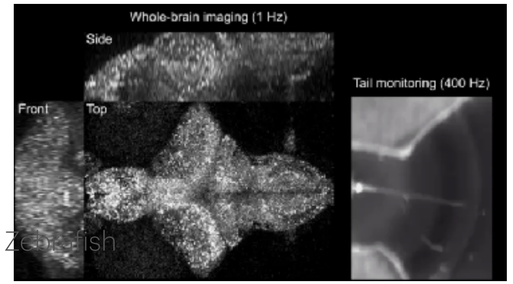


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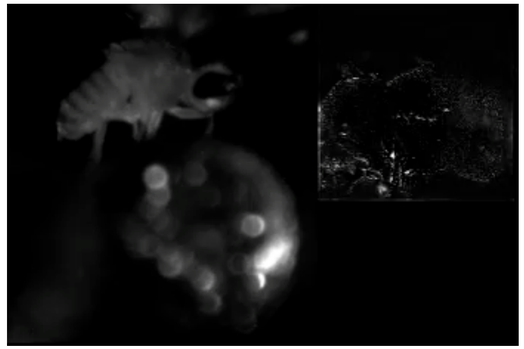


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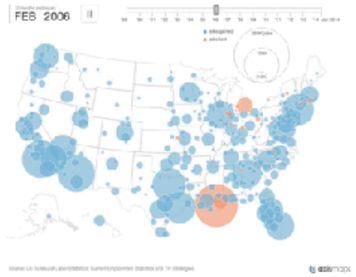


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Complex systems: a loose and incomplete definition



SCIENCE AND COMPLEXITY

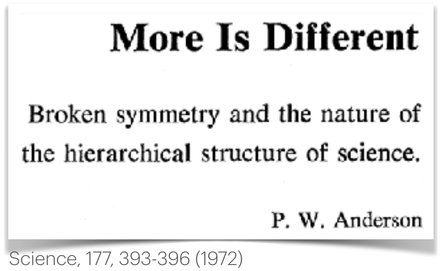
By WARREN WEAVER

Rockefeller Foundation, New York City

American Scientist, 36, 536-544 (1948)

statistical methods hold the key. They are all problems which involve dealing simultaneously with a *sizable number of factors which are interrelated into an organic whole.* They are all, in the language here proposed, problems of *organized complexity*.

On what does the price of wheat depend? This too is a problem of organized complexity. A very substantial number of relevant variables is involved here, and they are all interrelated in a complicated, but nevertheless not in helter-skelter, fashion.



More Is Different

Broken symmetry and the nature of the hierarchical structure of science.

P. W. Anderson

Science, 177, 393-396 (1972)

derstood, at least in principle. In this case we can see how the whole becomes not only more than but very different from the sum of its parts.

"constructionist" one: The ability to reduce everything to simple fundamental laws does not imply the ability to start from those laws and reconstruct the universe. In fact, the more the ele-

▷ large number of interacting elements

- neurons, people, cars, etc.

▷ nontrivial structure of interactions

- somewhere between regular and random

▷ emergence of new properties or behaviors

- the whole is more than the sum of the parts
- beyond the grasp of reductionism

Emergence of a “new” discipline

The idea of complex networks and complex systems is not new, and dates back to several decades, even centuries.

What explains the emergence of Network Science?

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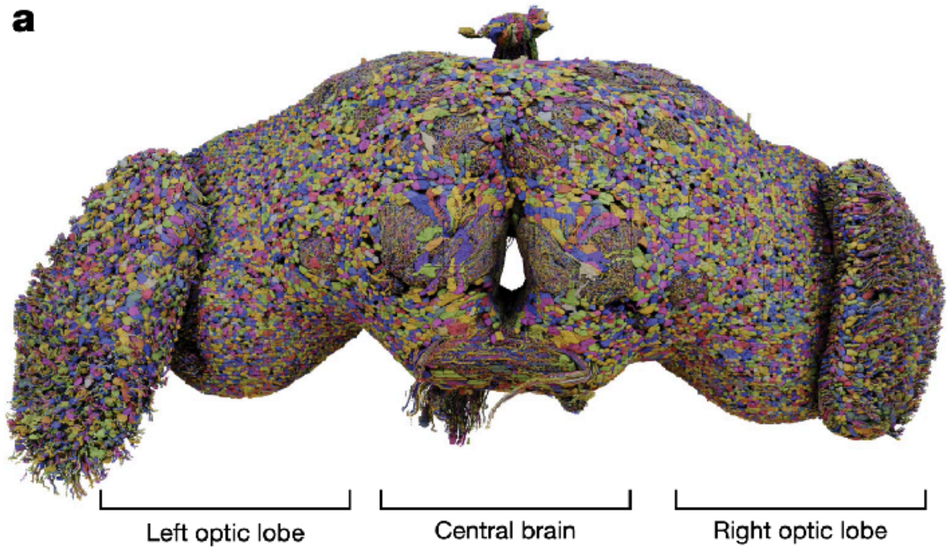
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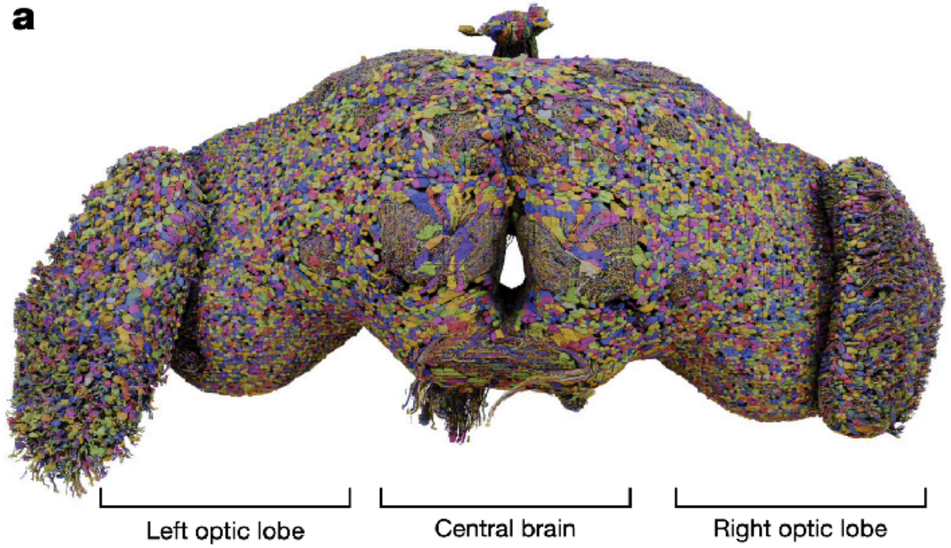
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What explains the emergence of Network Science?

- ▷ The digital revolution made possible the extraction of the wiring diagram of many complex systems.
- ▷ The universality of the architecture of these wiring diagrams across various domains.
  - → “universal” key organizational principles
  - → characterized using common measures
  - → tools, models and concepts “can be transferred” to other contexts
  - → common language to communicate across disciplines



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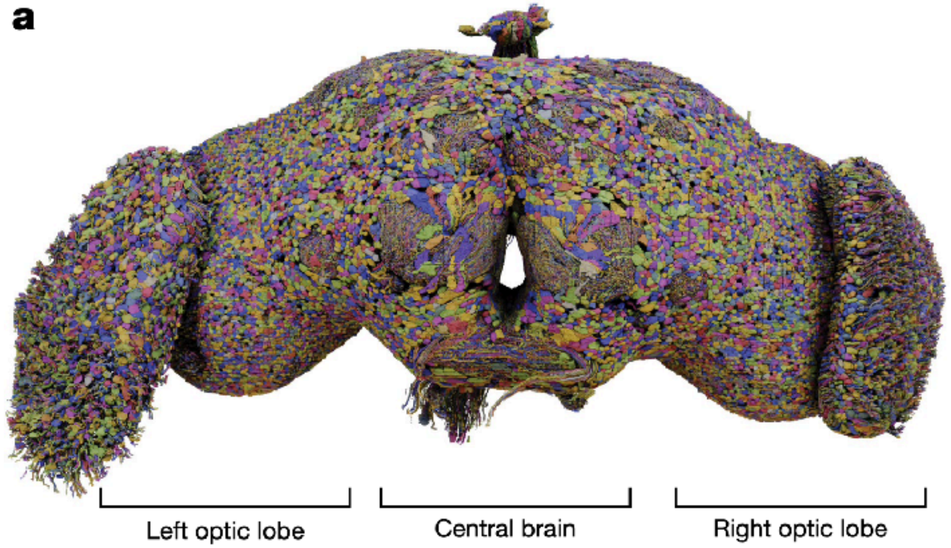
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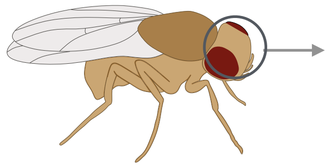
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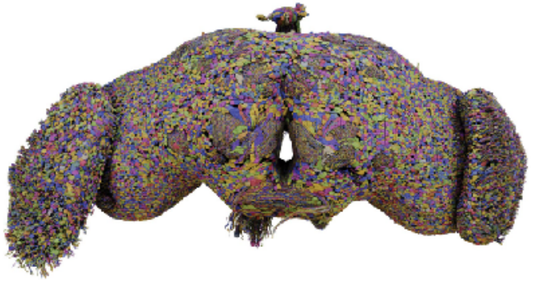
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Complex networks: A unifying paradigm to study and model complex systems

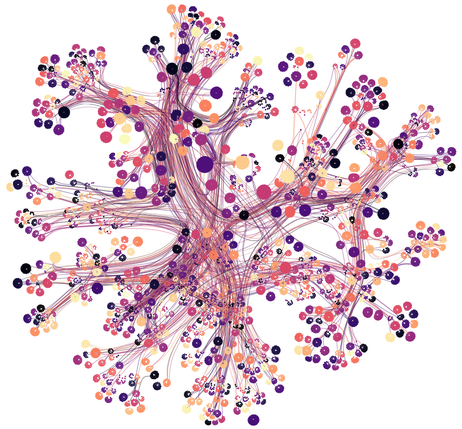
Mathematical modeling of complex systems





structure

complex network

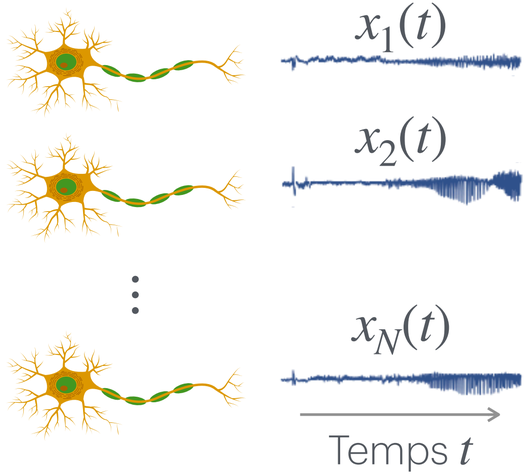


adjacency matrix

$$W = \begin{pmatrix} 0 & -1 & 0.5 & \dots & 0 \\ 0 & 0 & 2 & \dots & 1 \\ 3 & 1 & 0 & \dots & -5 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -0.2 & 0 & \dots & 0 \end{pmatrix}$$

dynamics





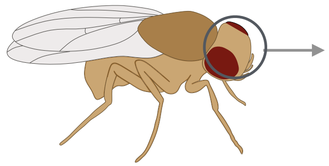
$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, \dots, x_N; W)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, \dots, x_N; W)$$
$$\vdots$$
$$\frac{\mathrm{d}x_N}{\mathrm{d}t} = f_N(x_1, \dots, x_N; W)$$

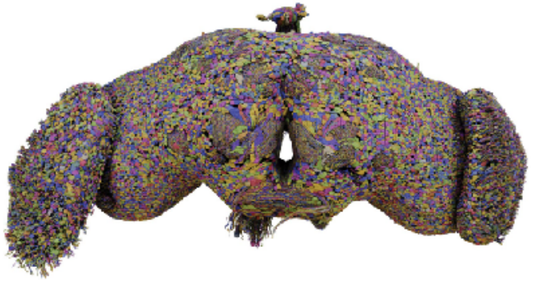
N : number of vertices/variables
(ex. number of neurons)

dynamical system

Scheffer et al., eLife, 2021
Aimon et al., eLife, 2023
Dorkenwald et al., Nature, 2024

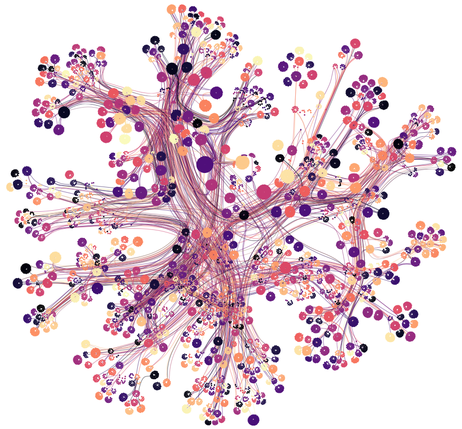
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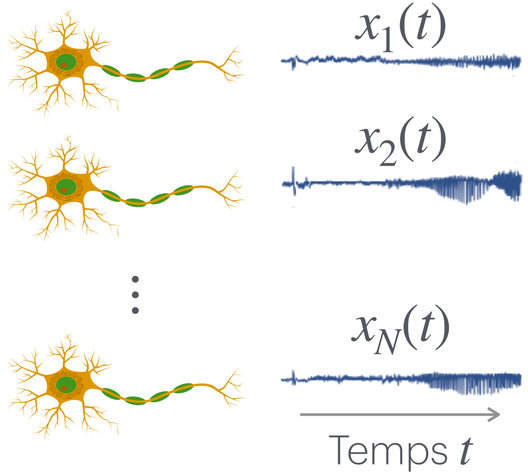


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dynamics





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Outline

1. Complex systems, complex networks and modelling

2. Are simple models enough to study complex systems/networks?

3. Simple encoding of structural complexity using latent metric space

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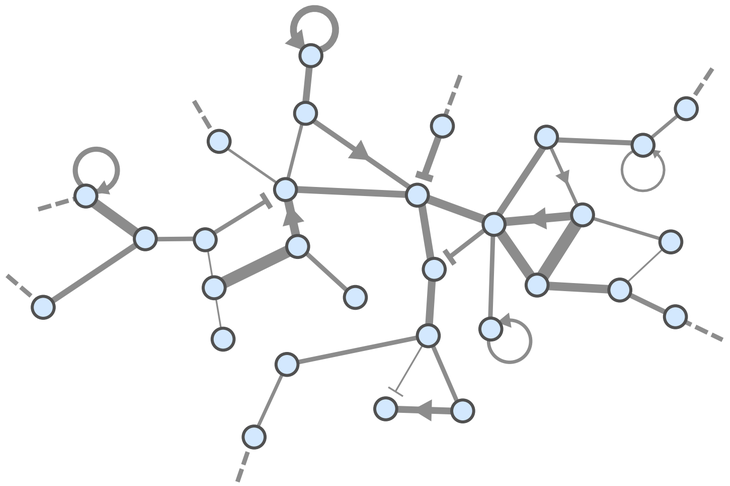
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Modeling the dynamics of complex systems

Complex network

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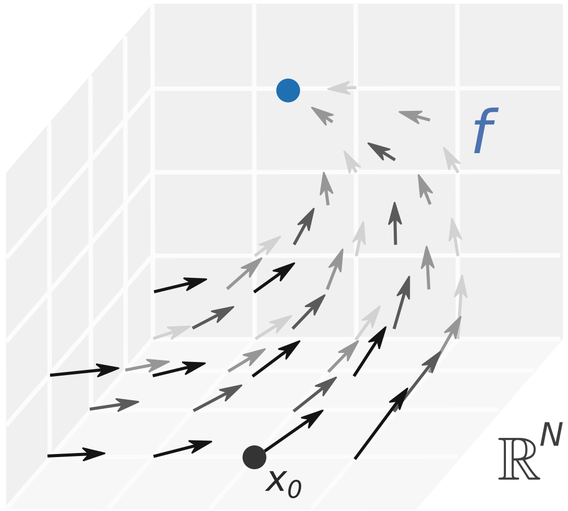


Vector field

f

High-dimensional dynamics

$\dot{x}=f(x;W)$



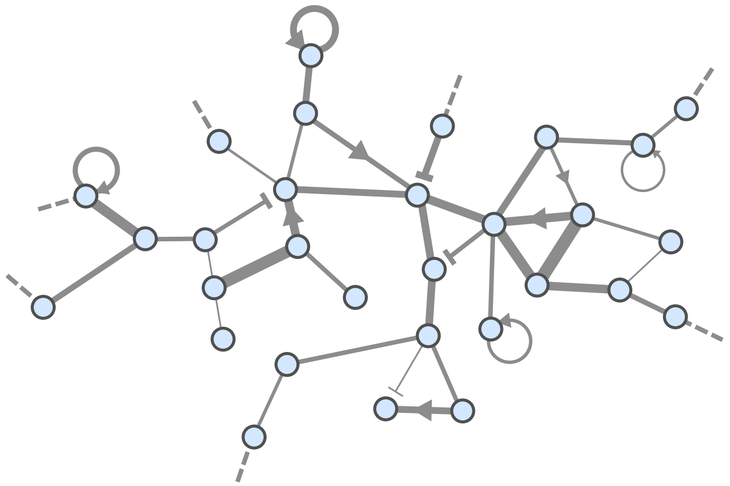
High-dimensional dynamics are hard to analyze and extracting meaningful intuitions from them is difficult.

Most mathematical models involve low-dimensional representations of the networks and/or dynamical observables.

Modeling the dynamics of complex systems

Complex network

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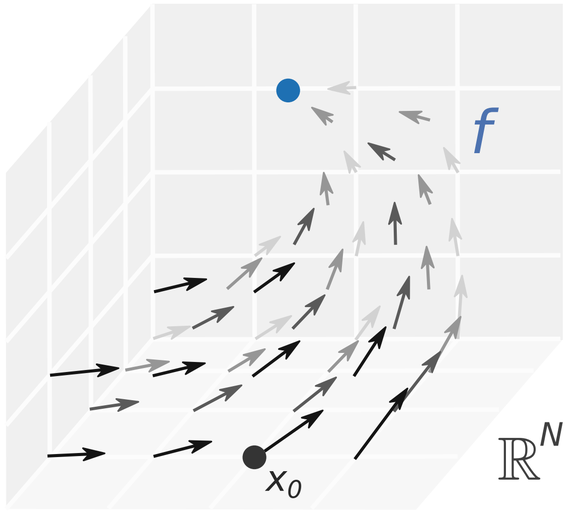


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Can we justify such a low-dimensional approach beyond mathematical and/or conceptual convenience?

The rank of adjacency matrices

Rank r of a matrix

- number of linearly independent rows/columns
- dimension of the vector space generated by its rows/columns
- number of nonzero singular values

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Singular value decomposition (SVD)



$\longrightarrow$ $W = U \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} V^\mathsf{T}$

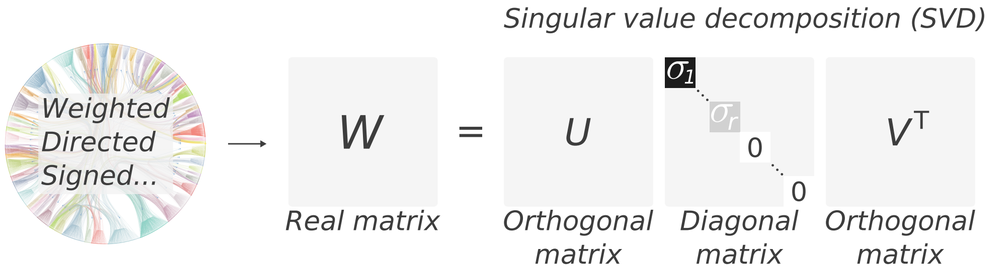
Real matrix | *Orthogonal matrix* | *Diagonal matrix* | *Orthogonal matrix*

$$\mathbf{W} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\mathsf{T} = \sigma_1 \begin{bmatrix} | \\ \mathbf{u}_1 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_1^\mathsf{T} & — \end{bmatrix} + \sigma_2 \begin{bmatrix} | \\ \mathbf{u}_2 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_2^\mathsf{T} & — \end{bmatrix} + \sigma_3 \begin{bmatrix} | \\ \mathbf{u}_3 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_3^\mathsf{T} & — \end{bmatrix} + \dots$$

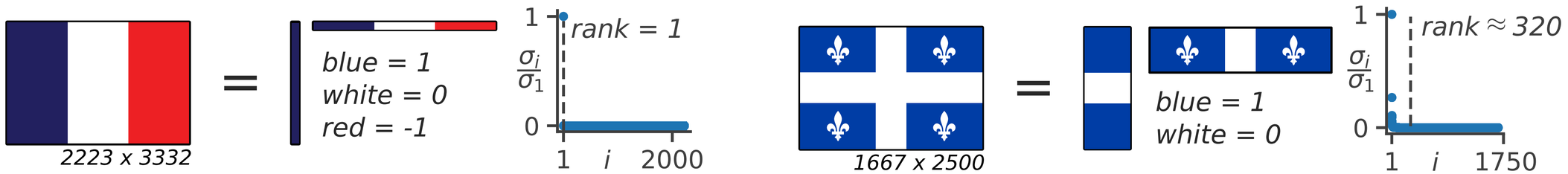
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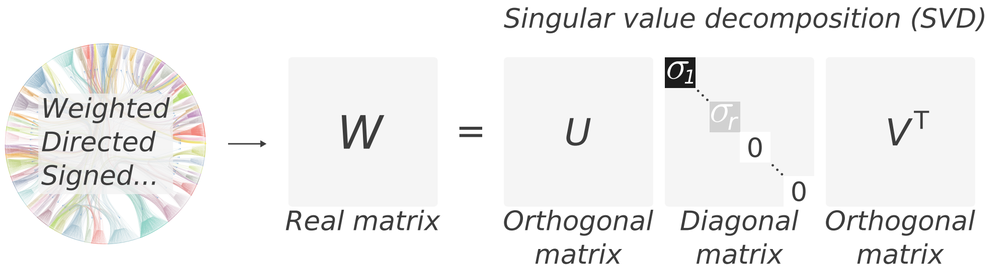
$$\mathbf{W} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\intercal = \sigma_1 \begin{bmatrix} | \\ \mathbf{u}_1 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_1^\intercal & — \end{bmatrix} + \sigma_2 \begin{bmatrix} | \\ \mathbf{u}_2 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_2^\intercal & — \end{bmatrix} + \sigma_3 \begin{bmatrix} | \\ \mathbf{u}_3 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_3^\intercal & — \end{bmatrix} + \dots$$



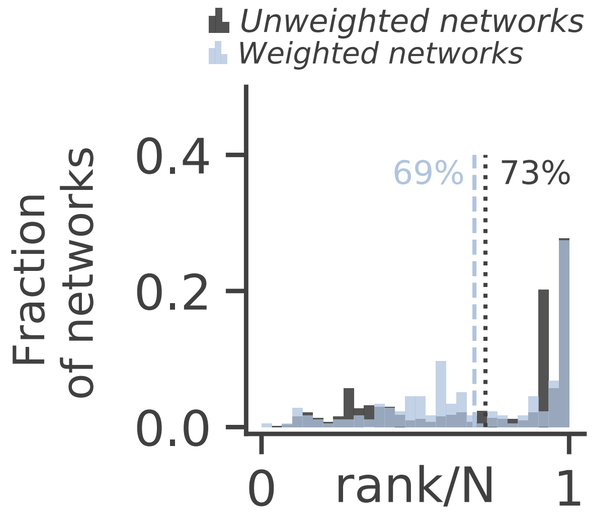
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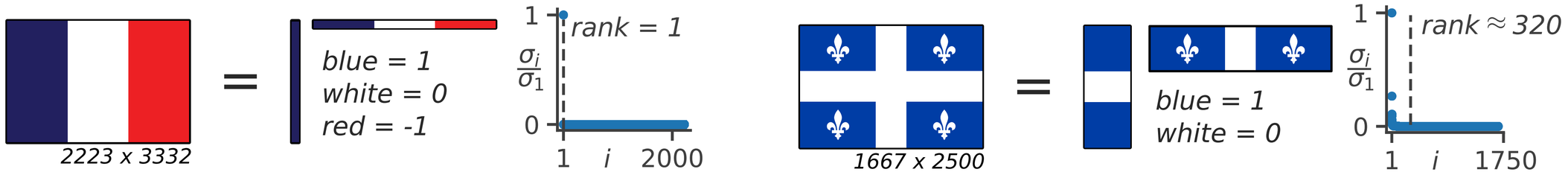


Full-rank (or almost full rank) adjacency matrices are very common!



Results for 679 empirical networks (502 unweighted networks and 177 weighted networks) downloaded from Netzschleuder.

$$\mathbf{W} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\intercal = \sigma_1 \begin{bmatrix} | \\ \mathbf{u}_1 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_1^\intercal & — \end{bmatrix} + \sigma_2 \begin{bmatrix} | \\ \mathbf{u}_2 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_2^\intercal & — \end{bmatrix} + \sigma_3 \begin{bmatrix} | \\ \mathbf{u}_3 \\ | \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_3^\intercal & — \end{bmatrix} + \dots$$

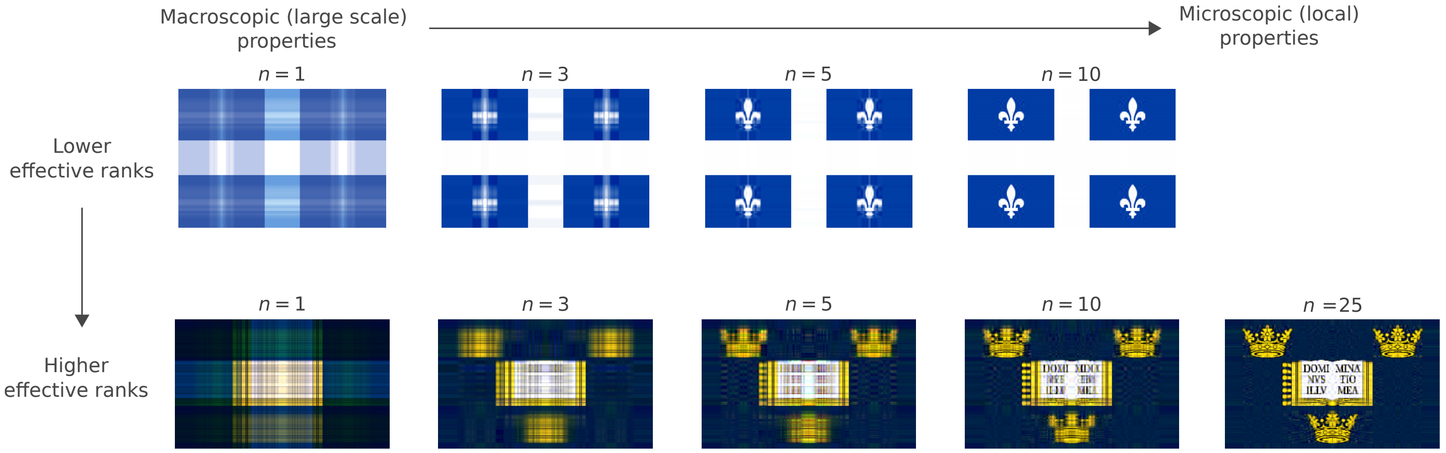


The effective ranks of adjacency matrices

$$\mathbf{W} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top \simeq \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$$

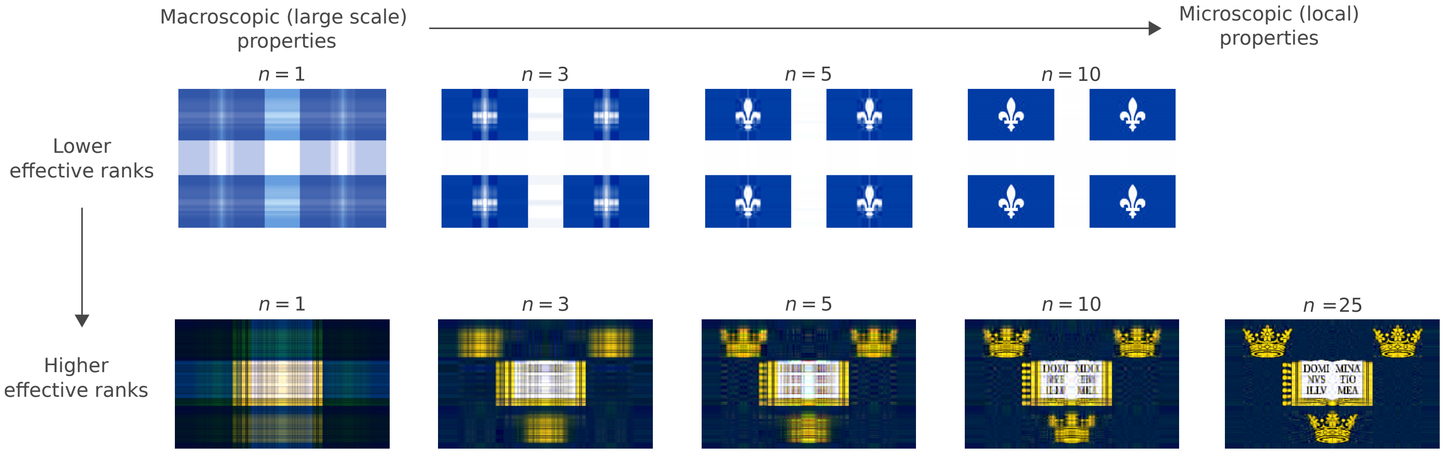
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Effective rank n of a matrix

- number of "significant" nonzero singular values
- term after which it is "reasonable" to truncate the sum

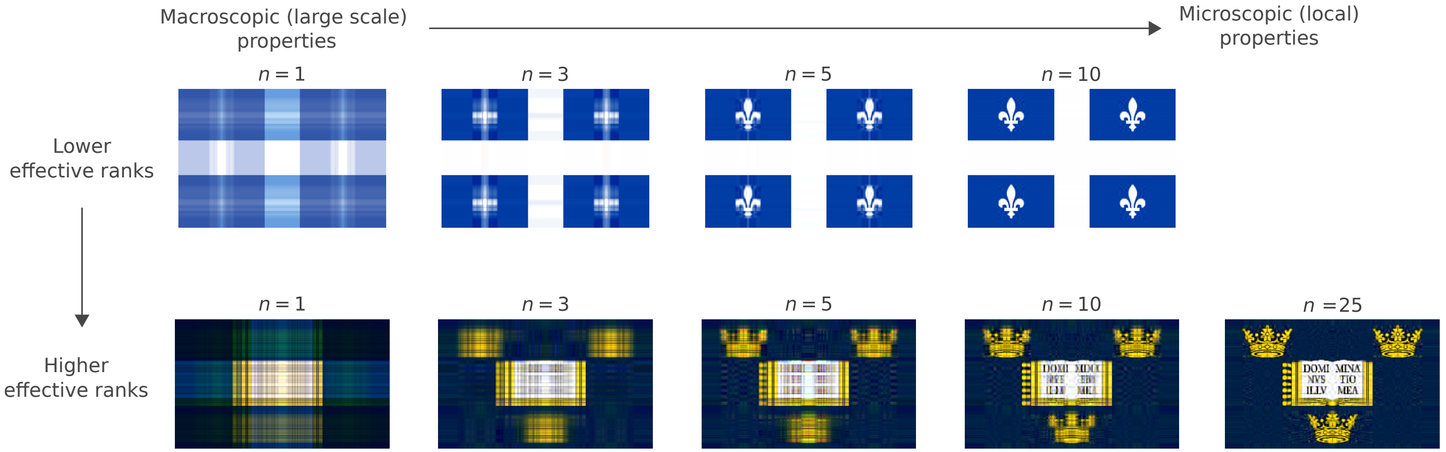
Abbreviation	Expression
srank	$\sum_{i=1}^{r} \sigma_i^2 / \sigma_1^2$
nrank	$\sum_{i=1}^{r} \sigma_i / \sigma_1$
energy	$\min\left[\arg\max_{\ell \in \{1,\ldots,N\}} \left(\sum_{i=1}^{\ell} \sigma_i^2 / \sum_{j=1}^{r} \sigma_j^2 > \tau\right)\right]$
elbow	$\frac{1}{\sqrt{2}} \arg\max_{i \in \{1,\ldots,N\}} \left\lvert \frac{i-1}{N-1} + \frac{\sigma_i - \sigma_N}{\sigma_1 - \sigma_N} - 1 \right\rvert - 1$
erank	$\exp\left\lvert -\sum_{i=1}^{r} \frac{\sigma_i}{\sum_{j=1}^{r} \sigma_j} \log \frac{\sigma_i}{\sum_{j=1}^{r} \sigma_j} \right\rvert$
thrank	$\#\left\{\sigma_i \mid i \in \{1, \ldots, N\} \text{ and } \sigma_i > \frac{4\sigma_{\text{med}}}{\sqrt{3}\mu_{\text{med}}}\right\}$
shrank	$\#\{s^*(\sigma_i) \mid i \in \{1, \ldots, N\} \text{ and } s^*(\sigma_i) > 0\}$

The effective ranks of adjacency matrices

$$\mathbf{W} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top \simeq \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$$

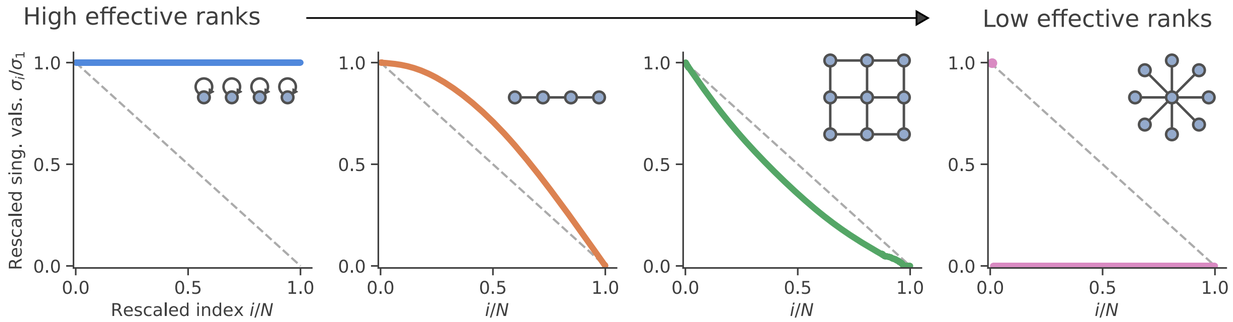
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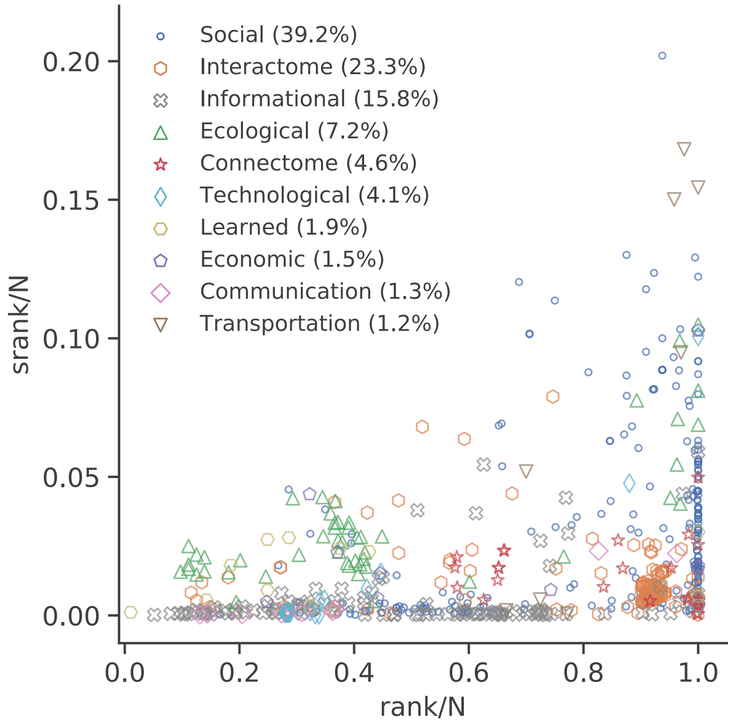
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The singular values (and the effective rank) encode information about the network's topology.



The effective ranks of adjacency matrices

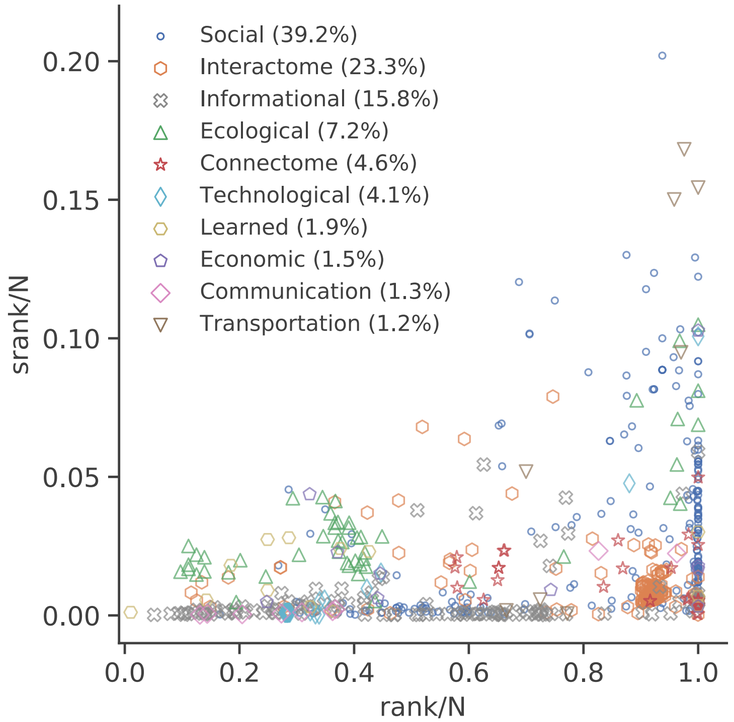
Many empirical networks appear to have a low effective rank!



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The effective ranks of adjacency matrices

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Results for 679 empirical networks (502 unweighted networks and 177 weighted networks) dowloaded from Netzschleuder.

But what does “low” mean?

A *workable* definition of "low" effective rank

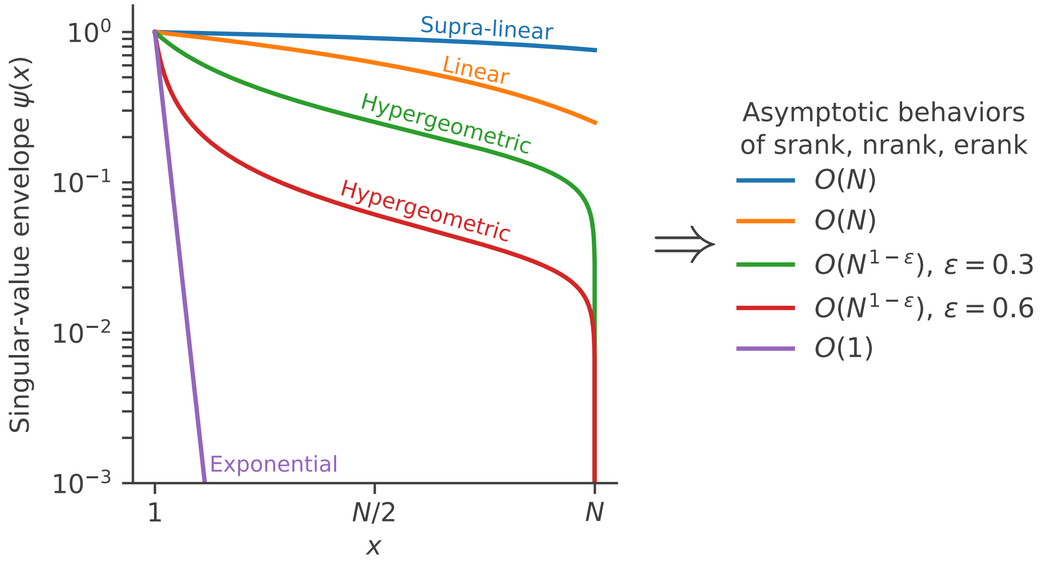
Hint: the rapid decrease of the dominant singular values of the adjacency matrix implies a low effective rank

- low effective rank? $\Rightarrow$ effective rank scales at most sublinearly as the number of nodes, N, goes to infitnity ($N^{1-\varepsilon}$ with $\varepsilon \in (0, 1]$)

A *workable* definition of "low" effective rank

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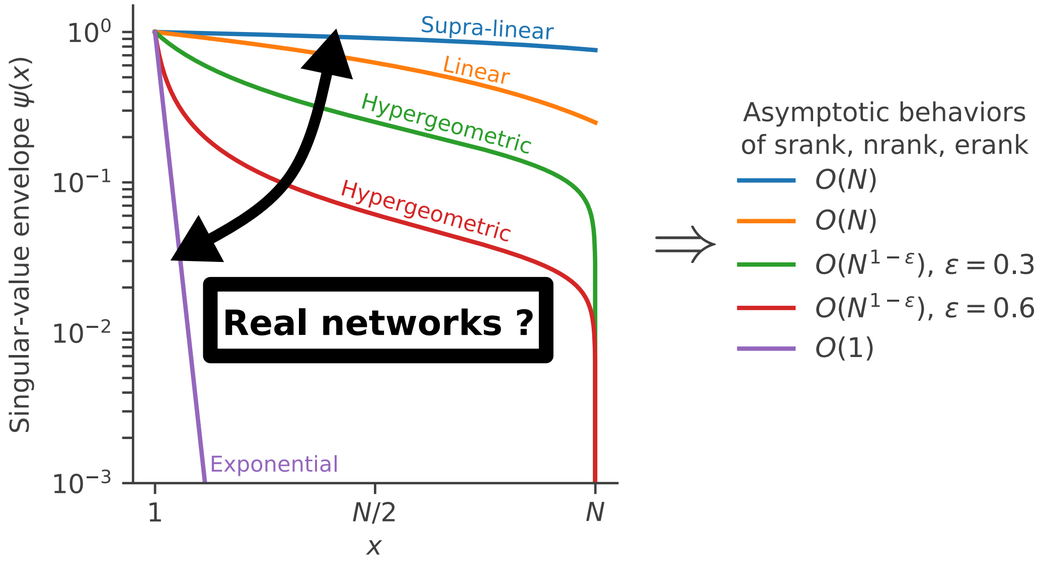
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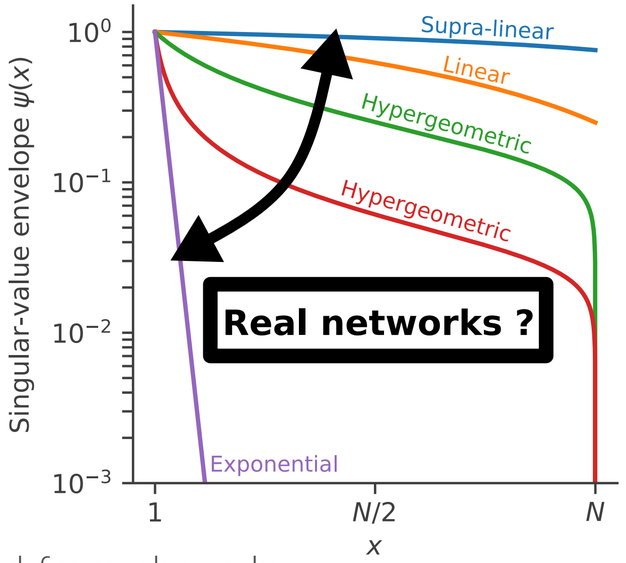
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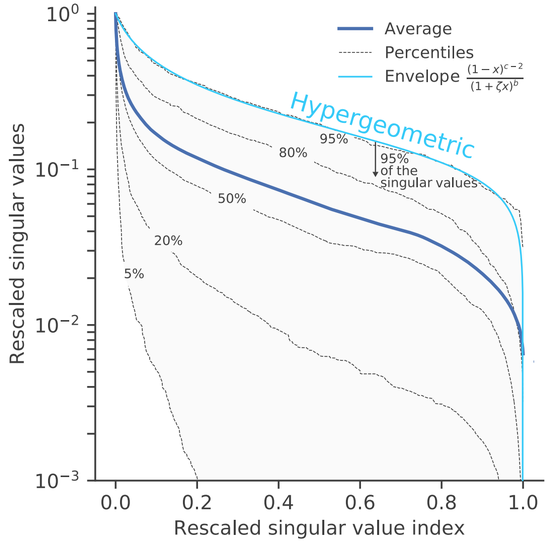


⇒

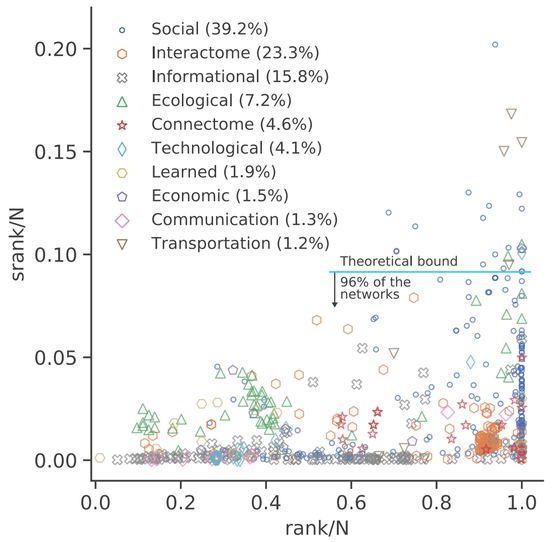
Asymptotic behaviors of srank, nrank, erank

- $O(N)$
- $O(N)$
- $O(N^{1-\varepsilon})$, $\varepsilon = 0.3$
- $O(N^{1-\varepsilon})$, $\varepsilon = 0.6$
- $O(1)$

The singular values of approx. 96% of considered networks are bounded from above by an hypergeometric envelope ⇒ sublinear effective ranks!



⇒



Workable definition of low effective rank: ∼10% of the number of nodes N

Approx. 96% of the 679 networks qualify for having a low effective rank!

The impact of effective low rank on the dynamics

Many dynamics on networks have the form

$$\dot{\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}, \boldsymbol{y})$$

with $\boldsymbol{x} \in \mathbb{R}^N$.

Examples:

- SIS (mean-field) : $\dot{x}_i = -d_i x_i + \gamma(1 - x_i)\, y_i$
- Wilson-Cowan: $\dot{x}_i = -d_i x_i + (1 - a x_i) \frac{1}{1+e^{-b(\gamma\, y_i - c)}}$
- Recurrent Neural Networks (RNN): $\dot{x}_i = -d_i x_i + \tanh(\gamma y_i + c_i)$
- Kuramoto-Sakaguchi: $\dot{z}_j = \mathrm{i}\omega_j z_j + \gamma\, e^{-\mathrm{i}\alpha}\, y_j - \gamma\, e^{\mathrm{i}\alpha}\, z_j^2\, \bar{y}_j$ with $z_j = e^{\mathrm{i}\theta_j}$
- Population dynamics: $\dot{x}_i = -d x_i + \gamma x_i y_i$ (Lotka-Volterra)
  $\dot{x}_i = -d x_i - s x_i^2 + \gamma \frac{x_i y_i}{\alpha + y_i}$
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for $i, j \in \{1, ..., N\}$ and $y_i = \sum_{j=1}^{N} W_{ij} x_j$.

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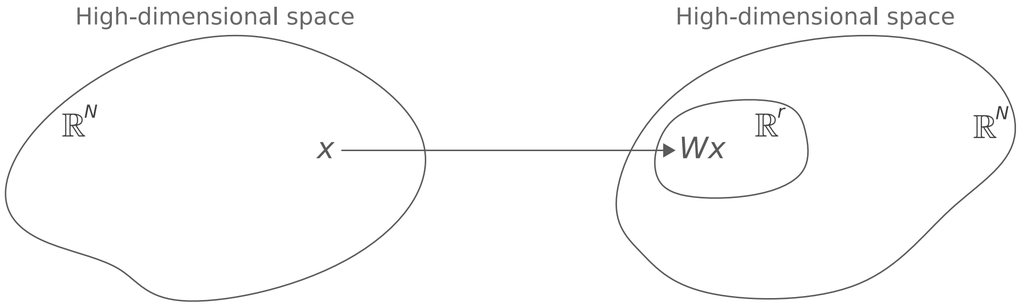
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Low rank $\mathbf{W} \Rightarrow \mathbf{W}\boldsymbol{x}$ belongs to a low-dimensional subspace



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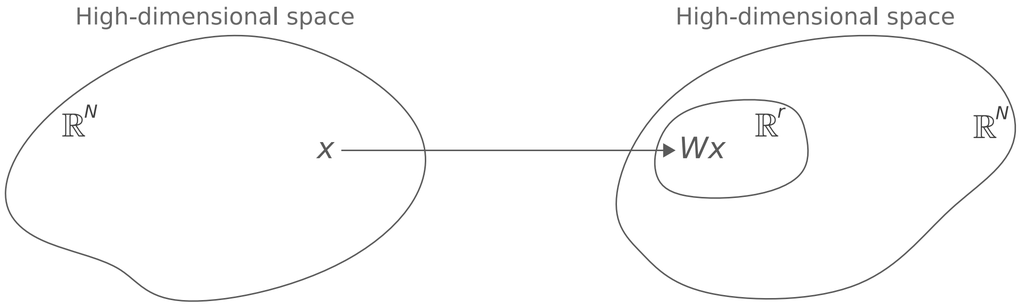
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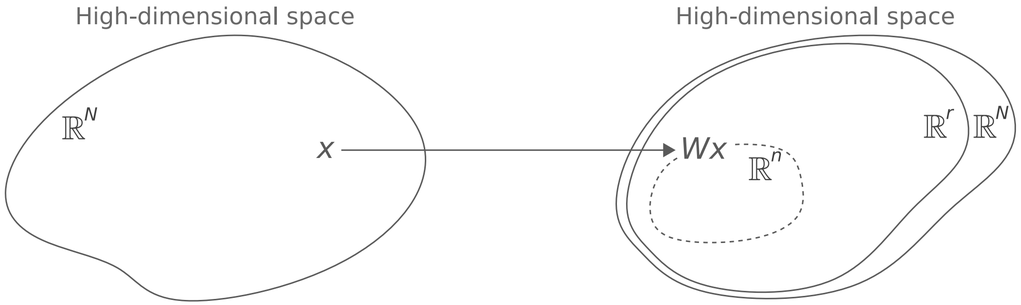
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for $i, j \in \{1, ..., N\}$ and $y_i = \sum_{j=1}^{N} W_{ij} x_j$.

Low rank $\mathbf{W} \Rightarrow \mathbf{W}\boldsymbol{x}$ belongs to a low-dimensional subspace



Low *effective* rank $\mathbf{W} \Rightarrow \mathbf{W}\boldsymbol{x}$ belongs to an *effectively* low-dimensional subspace



The impact of effective low rank on the dynamics

Original dynamics

$$\dot{\boldsymbol{x}} = \mathbf{g}(\boldsymbol{x}, \mathbf{W}\boldsymbol{x})$$

Reduced dynamics (with $\boldsymbol{X} = \mathbf{M}\boldsymbol{x}$)

$$\dot{\boldsymbol{X}} = \mathbf{M}\mathbf{g}(\mathbf{M}^{+}\mathbf{X}, \mathbf{W}\mathbf{M}^{+}\mathbf{X})$$

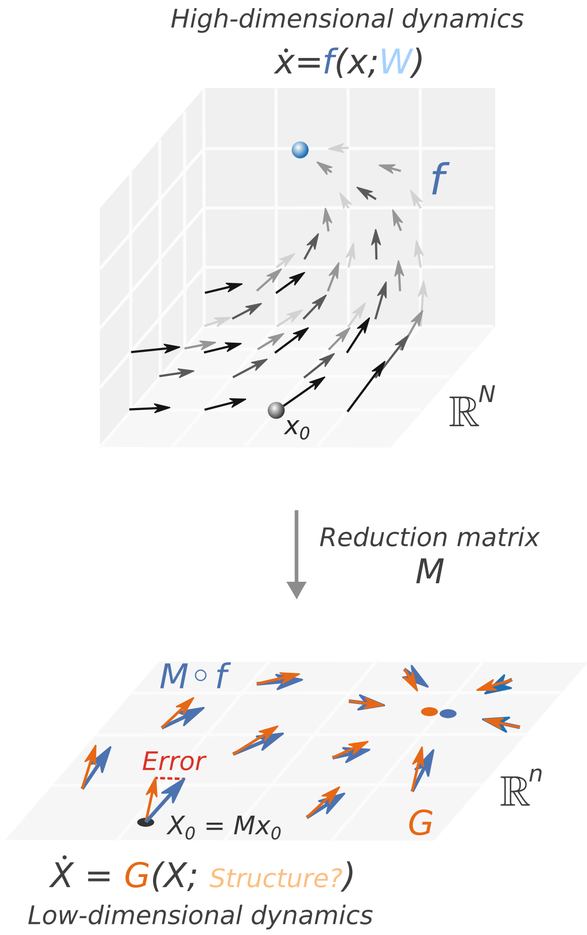
where n is the dimension of the reduced system
$\mathbf{M}^{+}$ denotes the pseudoinverse of $\mathbf{M}$
$\mathbf{M} = \mathbf{V}_n^{\top}$ is n-truncated right singular vector matrix

The alignment error is

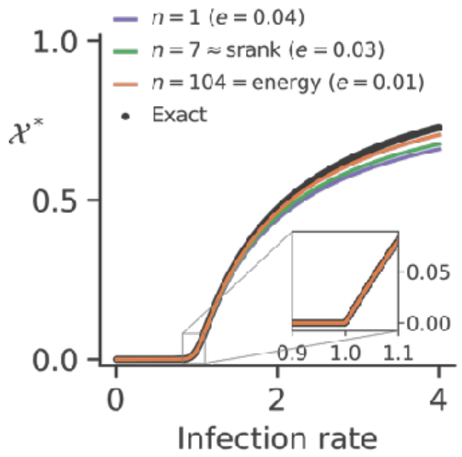
$$\mathcal{E}(x) \leq \frac{1}{\sqrt{n}}\left[\|\mathbf{V}_n^{\top}\mathbf{J}'_x(\mathbf{I} - \mathbf{V}_n\mathbf{V}_n^{\top})x\| + \sigma_{n+1}\|\mathbf{V}_n^{\top}\mathbf{J}'_y\|_2\|x\|\right]$$

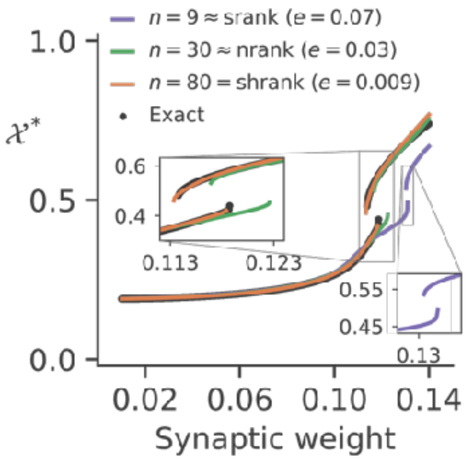
where $\mathbf{J}'_x, \mathbf{J}'_y$ are Jacobian matrices

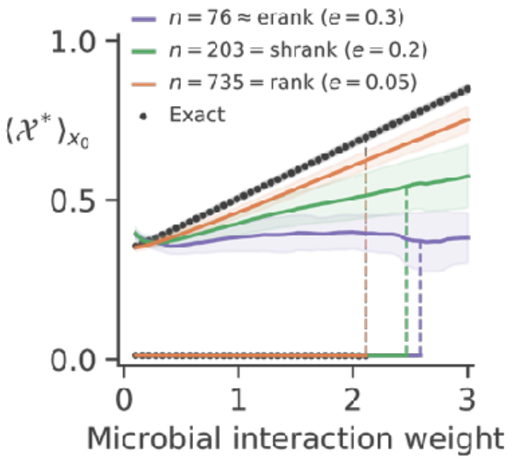
Rapid singular value decrease can induce
rapid alignment error decrease!

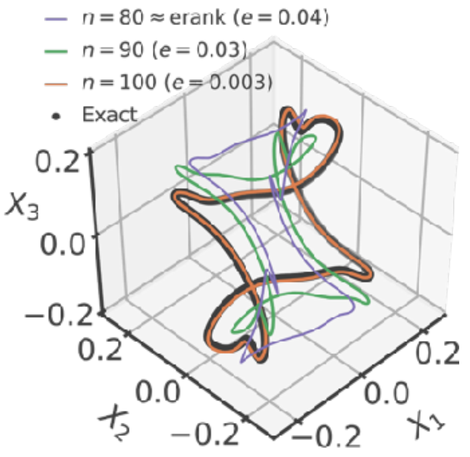


Reproduction of the dynamics with increasing accuracy

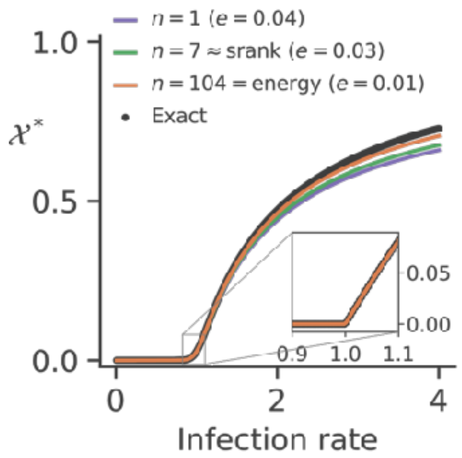


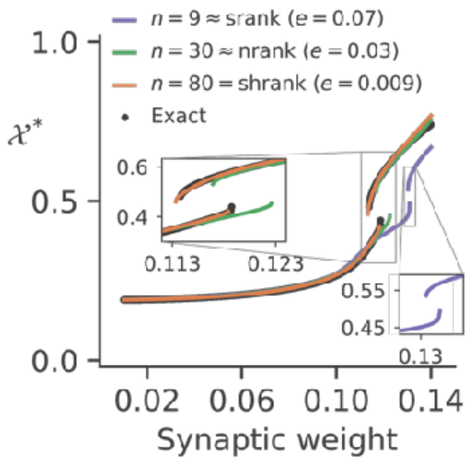


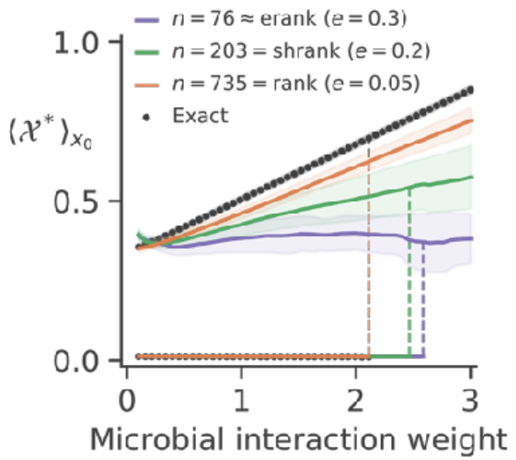


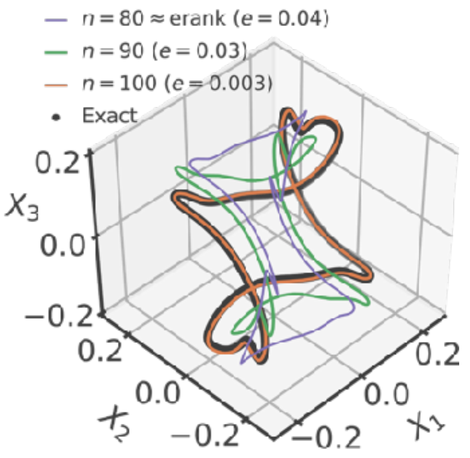


Reproduction of the dynamics with increasing accuracy









Emergence of higher-order interactions

Original Quenched mean-field SIS dynamics

$$\dot{x}_i = -\alpha x_i + \beta(1-x_i)\sum_{j=1}^{N} W_{ij}x_j, \quad i \in \{1,...,N\}$$

Reduced dynamics

$$\dot{X}_\mu = -\alpha X_\mu + \beta\sum_{\nu=1}^{n}\mathcal{W}_{\mu\nu}X_\nu - \beta\sum_{\nu,\tau=1}^{n}\mathcal{W}^{(3)}_{\mu\nu\tau}X_\nu X_\tau, \quad \mu \in \{1,...,n\}$$

Original Kuramoto-Sakaguchi dynamics

$$\dot{z}_j = i\omega_j z_j + \sum_{k=1}^{N} W_{jk}[z_k e^{-i\alpha} - z_j^2 \bar{z}_k e^{i\alpha}], \quad j \in \{1,...,N\}$$

Reduced dynamics

$$\dot{Z}_\mu = i\sum_{\nu=1}^{n}\Omega_{\mu\nu}Z_\nu + \sum_{\nu=1}^{n}\mathcal{W}^{(2)}_{\mu\nu}Z_\nu e^{-i\alpha} - \sum_{\alpha,\beta,\gamma=1}^{n}\mathcal{W}^{(4)}_{\mu\alpha\beta\gamma}Z_\alpha Z_\beta \bar{Z}_\gamma e^{i\alpha}$$


nature physics

PERSPECTIVE

The physics of higher-order interactions in complex systems

Federico Battiston, Enrico Amico, Alain Barrat, Ginestra Bianconi, Guilherme Ferraz de Arruda, Benedetta Franceschiello, Iacopo Iacopini, Sonia Kéfi, Vito Latora, Yamir Moreno, Micah M. Murray, Tiago P. Peixoto, Francesco Vaccarino and Giovanni Petri

Nat. Phys. 17, 1093 (2021)

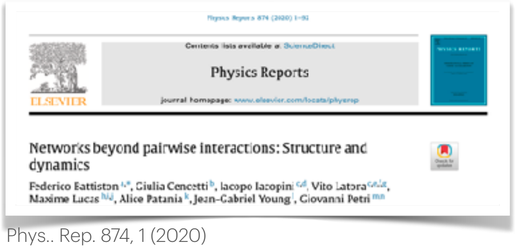

nature ecology & evolution

ARTICLES

Higher-order interactions capture unexplained complexity in diverse communities

Margaret M. Mayfield and Daniel B. Stouffer

Nat. Ecol. Evol. 1, 0062 (2017)


Physics Reports

Networks beyond pairwise interactions: Structure and dynamics

Federico Battiston, Giulia Cencetti, Iacopo Iacopini, Vito Latora, Maxime Lucas, Alice Patania, Jean-Gabriel Young, Giovanni Petri

Phys.. Rep. 874, 1 (2020)

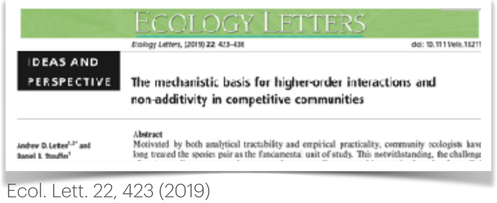

LETTER

doi:10.1038/nature23273

Higher-order interactions stabilize dynamics in competitive network models

Jacopo Grilli, György Barabás, Matthew J. Michalska-Smith & Stefano Allesina

Nature 548, 210 (2017)

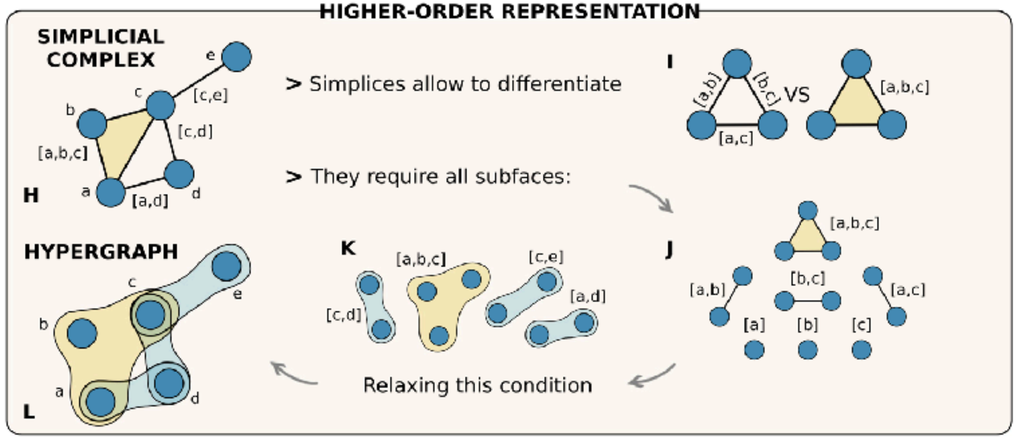

ECOLOGY LETTERS

IDEAS AND PERSPECTIVE

The mechanistic basis for higher-order interactions and non-additivity in competitive communities

Andrew D. Letten and Daniel B. Stouffer

Ecol. Lett. 22, 423 (2019)



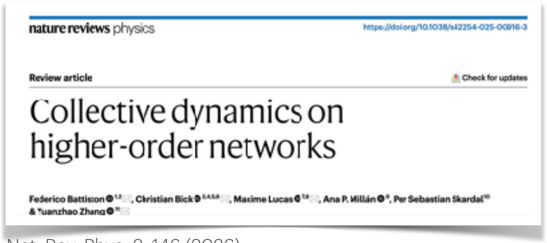

nature communications

Article

Reconstructing higher-order interactions in coupled dynamical systems

Federico Malizia, Alessandra Corso, Lucia Valentina Gambuzza, Giovanni Russo, Vito Latora & Mattia Frasca

Nat. Commun. 15, 5184 (2024)


nature reviews physics

Review article

Collective dynamics on higher-order networks

Federico Battiston, Christian Bick, Maxime Lucas, Ana P. Millán, Per Sebastian Skardal & Yuanzhao Zhang

Nat. Rev. Phys. 8, 146 (2026)


PNAS

RESEARCH ARTICLE | ECOLOGY

Coexistence in diverse communities with higher-order interactions

Theo Gibbs, Simon A. Levin and Jonathan M. Levine

PNAS 119, e2205063119 (2022)


nature communications

Article

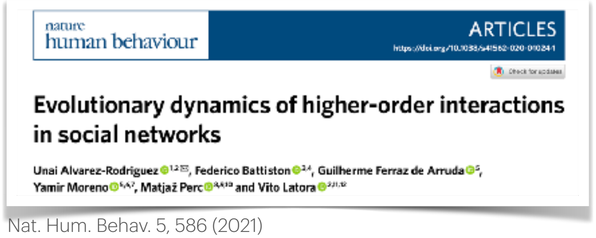
Characteristic scales and adaptation in higher-order contagions

Received: 7 October 2024

Accepted: 5 May 2025

Giulio Burgio, Guillaume St-Onge & Laurent Hébert-Dufresne

Nat. Commun. 16, 4589 (2025)

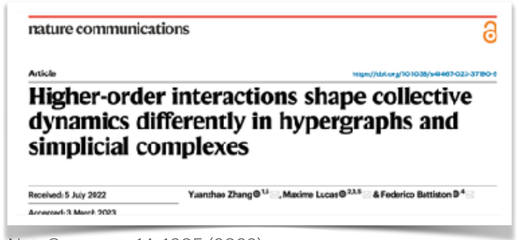

nature human behaviour

ARTICLES

Evolutionary dynamics of higher-order interactions in social networks

Unai Alvarez-Rodriguez, Federico Battiston, Guilherme Ferraz de Arruda, Yamir Moreno, Matjaž Perc and Vito Latora

Nat. Hum. Behav. 5, 586 (2021)


nature communications

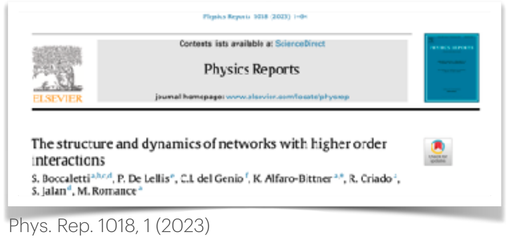
Article

Higher-order interactions shape collective dynamics differently in hypergraphs and simplicial complexes

Received: 5 July 2022

Yuanzhao Zhang, Maxime Lucas & Federico Battiston

Nat. Commun. 14, 1605 (2023)


Physics Reports

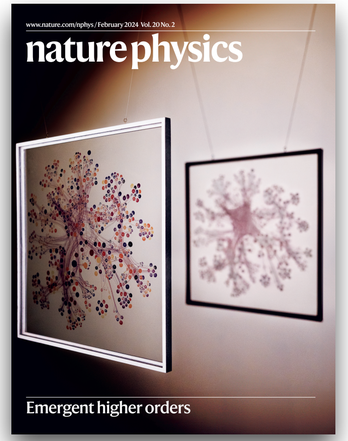
The structure and dynamics of networks with higher order interactions

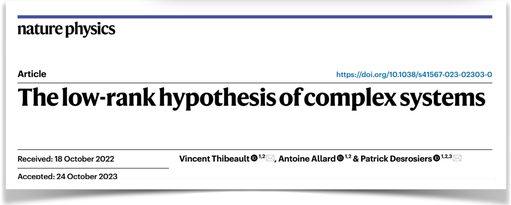
S. Boccaletti, P. De Lellis, C.I. del Genio, K. Alfaro-Bittner, R. Criado, S. Jalan, M. Romance

Phys. Rep. 1018, 1 (2023)

Main takeaways

- The rapid decrease of the singular values of adjacency matrices (i.e. low effective rank) offers a *justification* for low-dimensional mathematical models beyond mathematical and/or conceptual convenience.
- A large proportion of real networks can be considered as having a low effective rank.
- The higher-order interactions observed in some systems could be a byproduct of a low-dimensional representation used to analyze them.




nature physics
Article
https://doi.org/10.1038/s41567-023-02303-0

The low-rank hypothesis of complex systems

Received: 18 October 2022
Accepted: 24 October 2023

Vincent Thibeault, Antoine Allard & Patrick Desrosiers

Challenges and open questions

- Could we measure the effective dimension independently?
- Could we design a random graph model based on observed singular values (singular vectors)?
- Are some of the higher-order interactions inferred from time series artifacts of coarse-grained observations?
- Could we designed more interpretable observables, perhaps nonlinear ones?

Kuramoto meets Koopman: Constants of motion, symmetries, and network motifs

Vincent Thibeault[1,2*], Benjamin Claveau[1,2], Antoine Allard[1,2], Patrick Desrosiers[1,2,3]

arXiv:2504.06248

Operator-theoretic approach to the partial integration of randomly coupled phase oscillators

Vincent Thibeault,[1,2,*] Benjamin Claveau,[1,2] Antoine Allard,[1,2] and Patrick Desrosiers[1,2,3]

arXiv:2607.02617

An inverse spectral problem for dynamics on complex networks

Benjamin Claveau,[1,2,*] Vincent Thibeault,[1,2] Antoine Allard,[1,2] and Patrick Desrosiers[1,2,3]

Preprint on arXiv expected fall 2026



Benjamin Claveau
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Vincent Thibeault
Yale University

Patrick Desrosiers
Université Laval
CERVO Brain research center

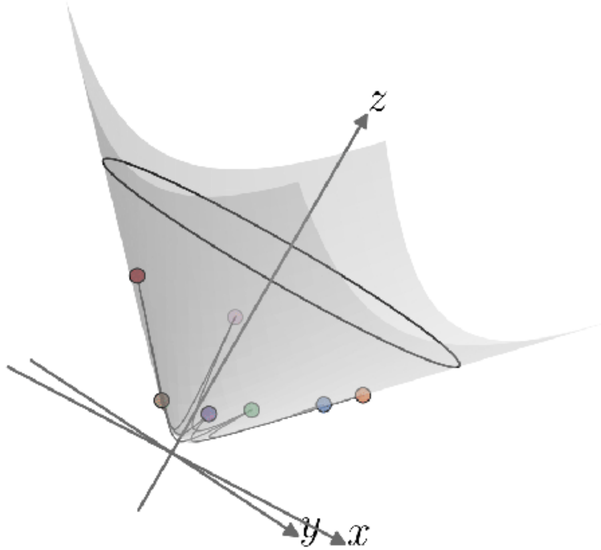


Outline

1. Complex systems, complex networks and modelling

2. Are simple models enough to study complex systems/networks?

3. Simple encoding of structural complexity with latent metric space

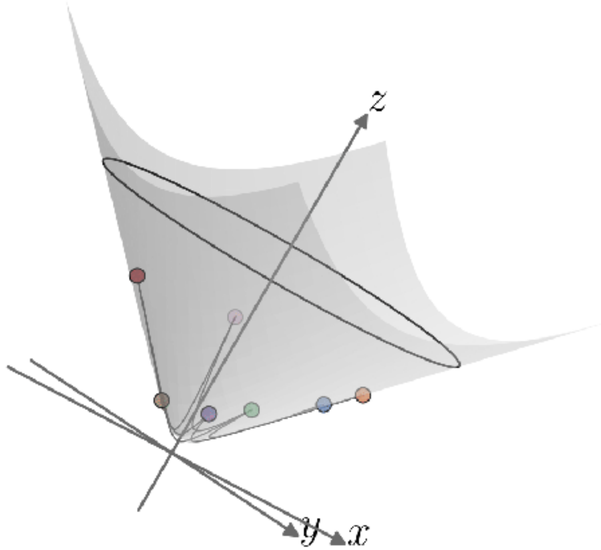


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1. Complex systems, complex networks and modelling

2. Are simple models enough to study complex systems/networks?

3. Simple encoding of structural complexity with latent metric space



Maximally random graph ensembles

The probability, $P(\mathbb{A})$, for a $N \times N$ adjacency matrix $\mathbb{A} = \{a_{ij}\} \in [0,1]^{\binom{N}{2}}$ that maximizes the entropy subjected to the L constraints ($l = 1, 2, \ldots, L$)

$$S(\{\mathbb{A}\}) = -\sum_{\mathbb{A}} P(\mathbb{A}) \ln P(\mathbb{A}) \qquad \bar{F}_l = \sum_{\mathbb{A}} F_l(\mathbb{A}) P(\mathbb{A})$$

is (α_l being the l-th Lagrange multiplier)

$$P(\mathbb{A}) \propto \exp\left(-\sum_{l=1}^{L} \alpha_l F_l(\mathbb{A})\right) .$$

Phys. Rev. E 70, 066117 (2004)
EPL 81, 28005 (2007)
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Example 1: fixing the expected number of edges

$$\bar{F}_1 = \sum_{i=1}^{N} \sum_{j=i+1}^{N} a_{ij} = M$$

yields the Bernouilli random graph ensemble

$$P(\mathbb{A}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} p^{a_{ij}} (1-p)^{1-a_{ij}} \quad \text{with} \quad p = \frac{1}{e^{\alpha_1} + 1} .$$

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Example 2: fixing the expected degree sequence

$$\bar{F}_l = \sum_{j=1}^{N} a_{lj} = k_l$$

for $l = 1, \ldots, N$ yields the soft configuration model

$$P(\mathbb{A}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} p_{ij}^{a_{ij}} (1-p_{ij})^{1-a_{ij}} \quad \text{with} \quad p_{ij} = \frac{1}{e^{-(\alpha_i+\alpha_j)} + 1} .$$

Redefining $\kappa_l = \sqrt{\langle\kappa\rangle N} e^{\alpha_l}$ for $l = 1, \ldots, N$ yields the Chung-Lu model

$$p_{ij} = \frac{1}{1 + \frac{\langle\kappa\rangle N}{\kappa_i \kappa_j}} \simeq \frac{\kappa_i \kappa_j}{\langle\kappa\rangle N} .$$

Studying the conditions for which p_{ij} can be factorized informs us on the degree-degree correlations observed in real networks.

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Modeling clustering

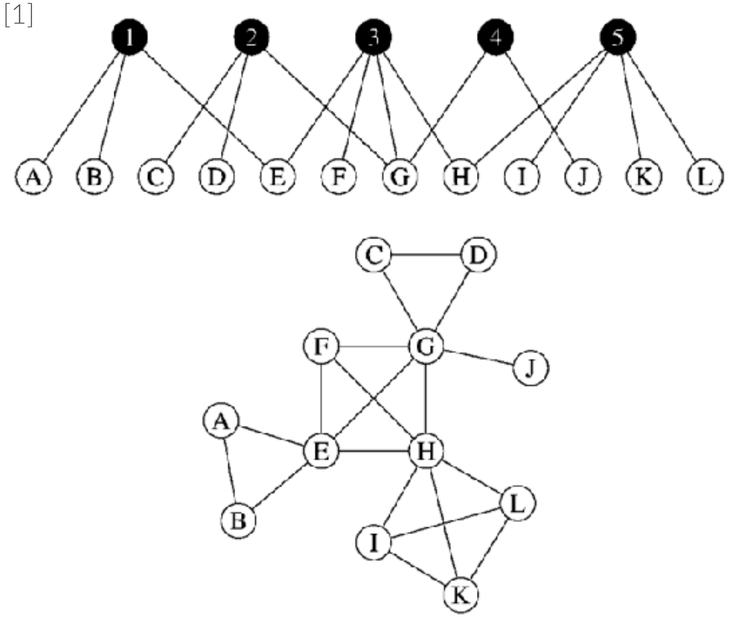
Tricky because clustering consists in three-node interactions while our mathematical tools rely on pairwise interactions either explicitly or implicitly.

Straightforward inclusion of triangles to the maximally random graph ensemble formalism yields unwanted behavior (ex.: triangle agglutination in the Strauss model [6]).

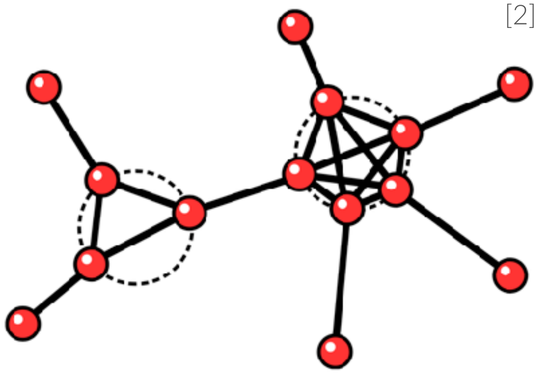
Most models therefore assume

- ▷ an underlying tree-like structure
- ▷ that the networks are dense

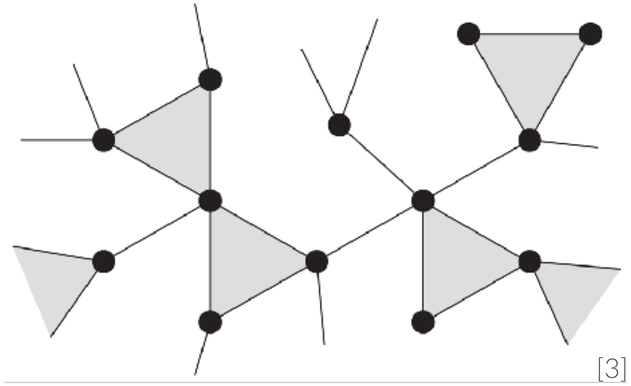
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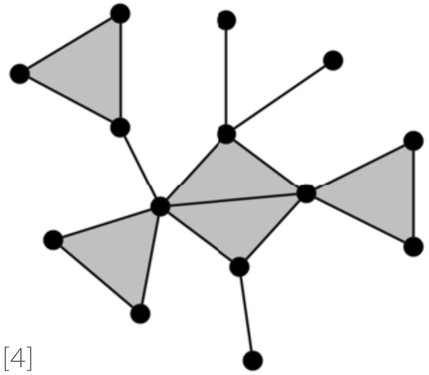
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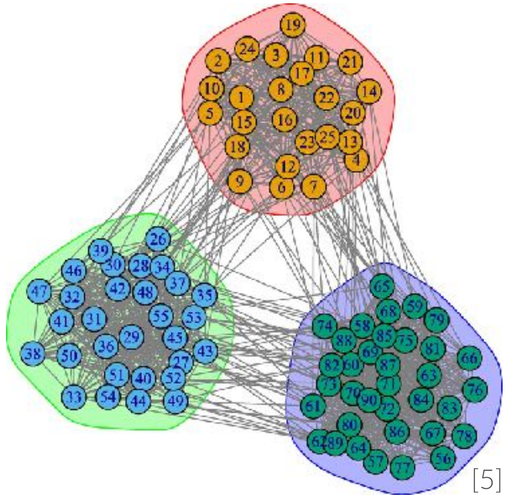
[3]



[4]



[5]



[1] Phys. Rev. E 68, 026121 (2003)
[2] Phys. Rev. E 80, 036107 (2009)
[3] Phys. Rev. Lett. 103, 058701 (2009)
[4] Phys. Rev. E 82, 066118 (2010)
[5] Appl. Netw. Sci. 4, 122 (2019)
[6] Phys. Rev. E 72, 026136 (2005)

Modeling clustering

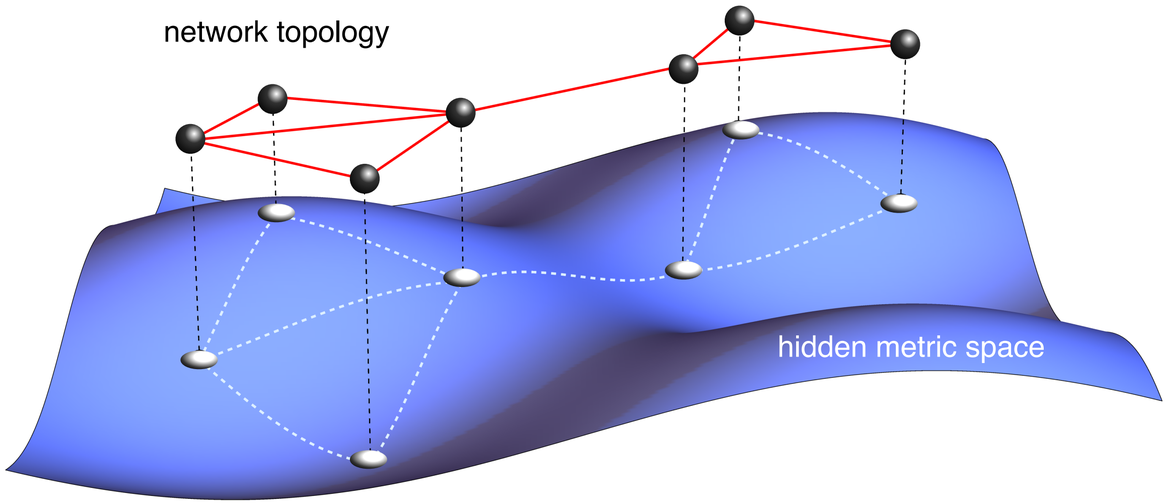
Idea: Assume that the vertices are embedded in a metric space and that any two vertices are connected with a probability that is a decreasing function of the distance between them.

Modeling clustering

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Properties of any metric space:

Identity of indiscernibles	$d(x,y) = 0 \quad \Leftrightarrow \quad x = y$
Non-negativity	$d(x,y) \geq 0$
Symmetry	$d(x,y) = d(y,x)$
Triangle inequality	$d(x,y) \leq d(x,z) + d(z,y)$



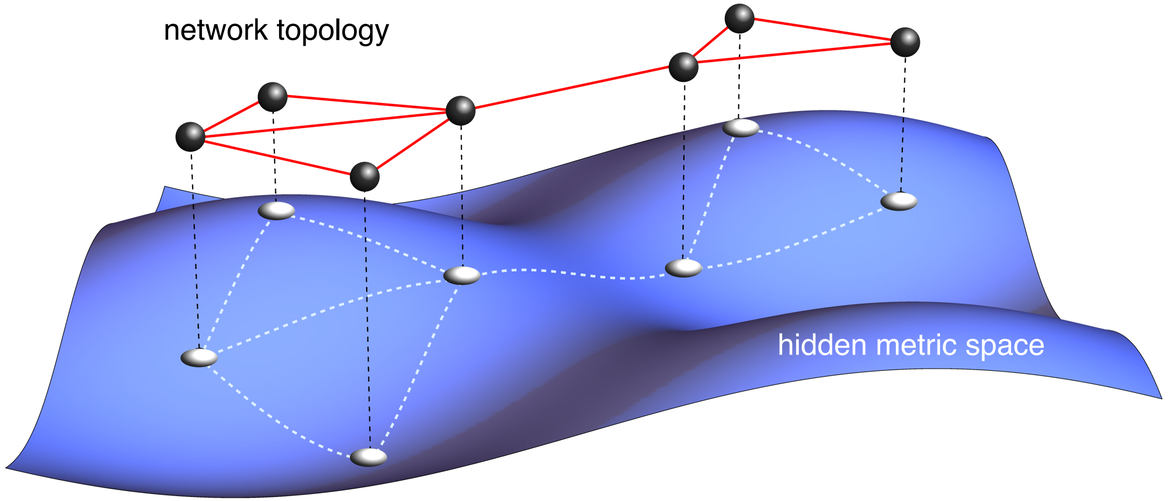
Courtesy of M. Boguñá

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Courtesy of M. Boguñá

Henceforth we consider that there is a distance matrix $\mathbb{X} = \{x_{ij}\}$ where x_{ij} is the distance between vertices i and j in the D-dimensional space.

Maximally random geometric graph ensembles

Example 3: fixing the expected number of edges and the expected total energy

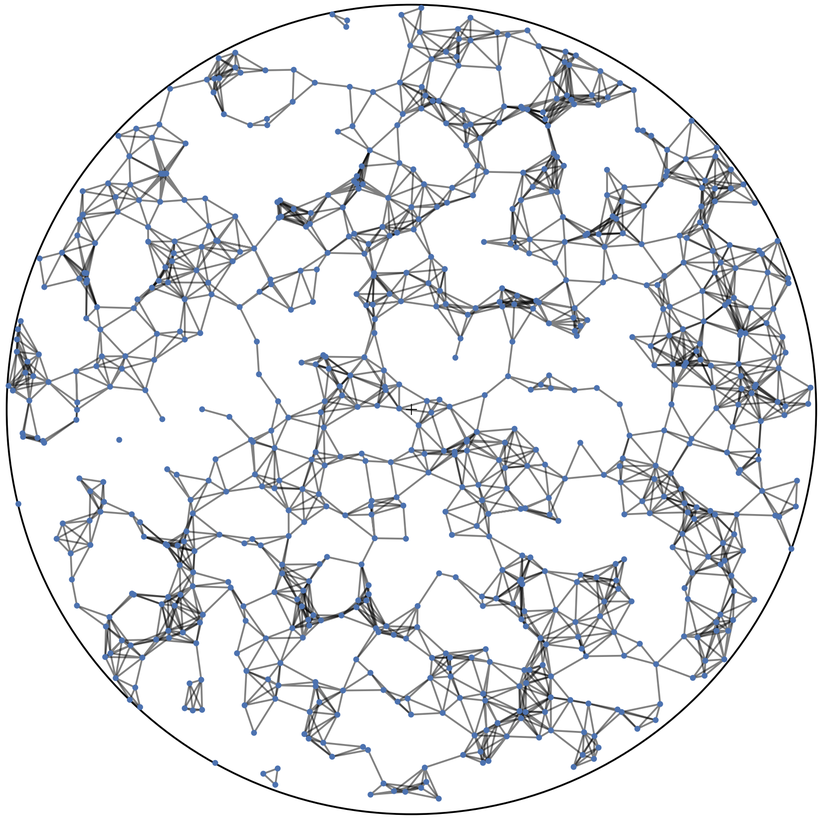
$$\bar{F}_1 = \sum_{i=1}^{N} \sum_{j=i+1}^{N} a_{ij} = M$$

$$\bar{F}_2 = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \varepsilon_{ij} a_{ij} = \sum_{i=1}^{N} \sum_{j=i+1}^{N} f(x_{ij}) a_{ij} = E$$

yields the homogeneous random geometric graph ensemble

$$P(\mathbb{A}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} p_{ij}^{a_{ij}} (1 - p_{ij})^{1-a_{ij}} \quad \text{with} \quad p_{ij} = \frac{1}{e^{\beta(\varepsilon_{ij} - \mu)} + 1} \, .$$

The graphs will be sparse, highly clustered and small-world iif $f(x_{ij}) \sim \ln x_{ij}$ and $\beta \in [D, D+2]$.



Phys. Rev. Research 2, 023040 (2020)
Phys. Rev. E 66, 016121 (2002)

Maximally random geometric graph ensembles

Example 4: fixing the expected degree sequence and the expected total energy

$$\bar{F}_l = \sum_{j=1}^{N} a_{lj} = k_l \quad (l = 1, \ldots, N)$$

$$\bar{F}_{N+1} = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \varepsilon_{ij} a_{ij} = \sum_{i=1}^{N} \sum_{j=i+1}^{N} f(x_{ij}) a_{ij} = E$$

yields the heterogeneous random geometric graph ensemble

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The graphs will be sparse, highly clustered, small-world and devoid of non-structural degree-degree correlation iif $f(x_{ij}) = \ln x_{ij}$ and $\beta \in [D, D+2]$[a]. Redefining $\alpha_l = -(\beta/D) \ln(\sqrt{\mu} \kappa_l)$ yields

$$p_{ij} = \frac{1}{e^{\beta(\varepsilon_{ij} - \mu)} + 1} \quad \text{with} \quad \varepsilon_{ij} = \ln \left(\frac{x_{ij}}{(\kappa_i \kappa_j)^{\frac{1}{D}}} \right) \, .$$

[a] No upper bound if expected degree sequence is scale-free.

Phys. Rev. E 80, 035101 (2009)
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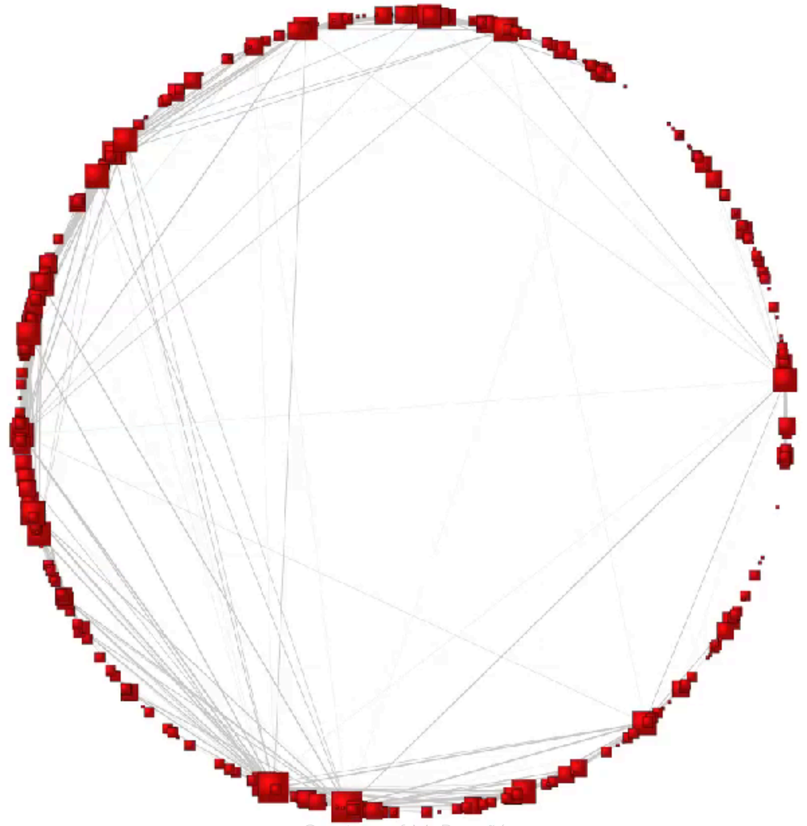
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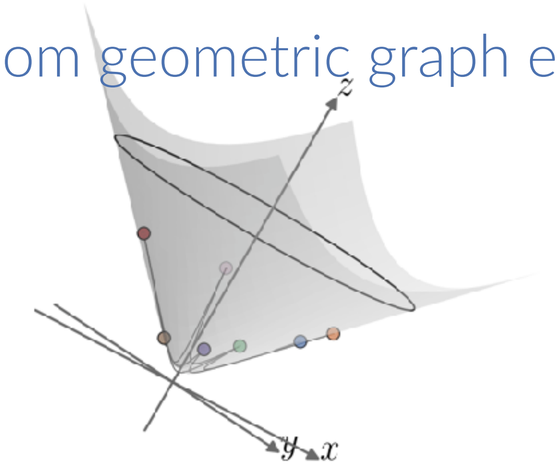
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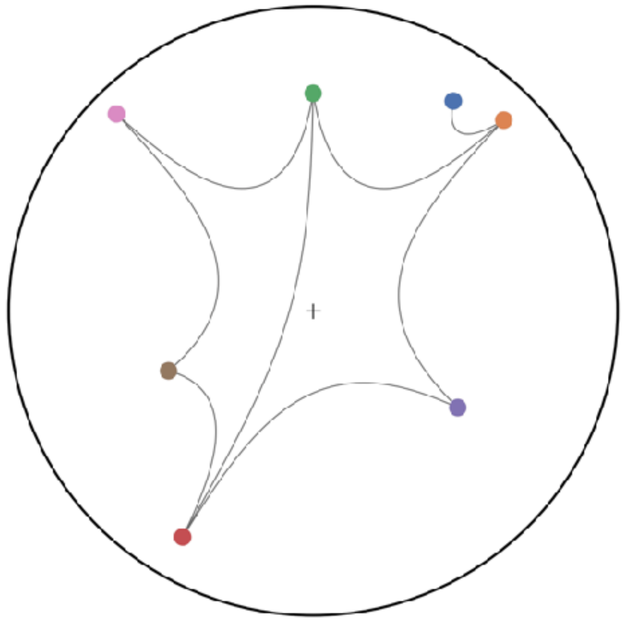
Courtesy of M. Boguñá

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Phys. Rev. Research 2, 023040 (2020)

Maximally random geometric graph ensembles

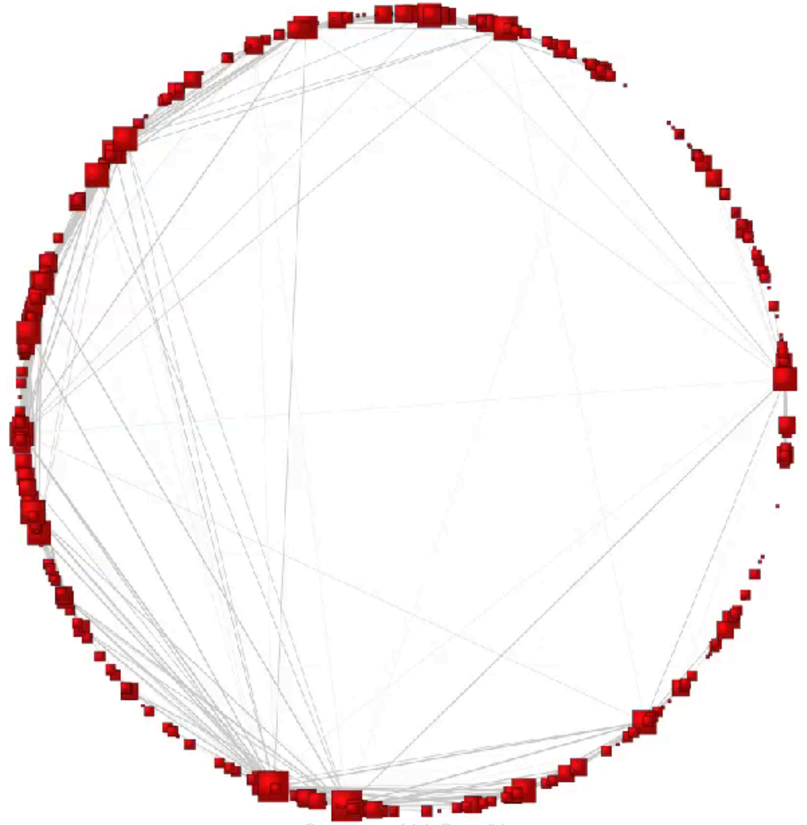


hyperboloid in $\mathbb{R}^{2,1}$



hyperbolic disk (r, θ)

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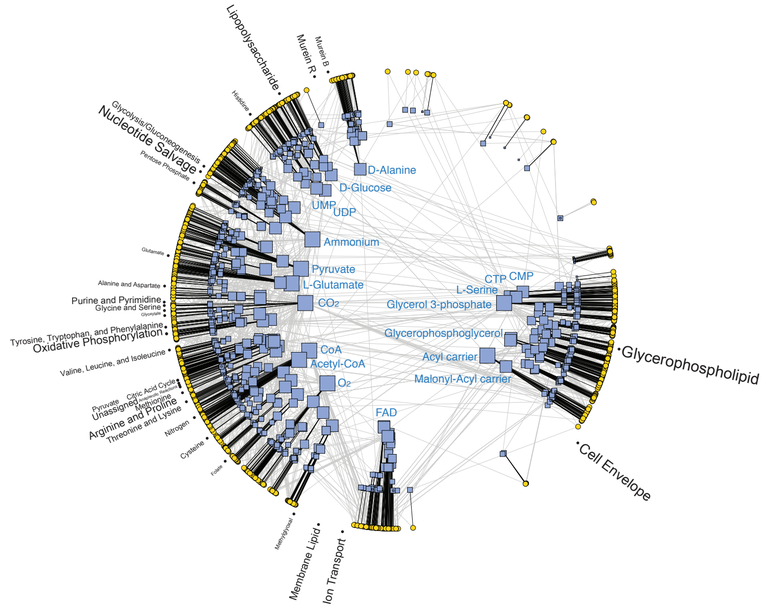


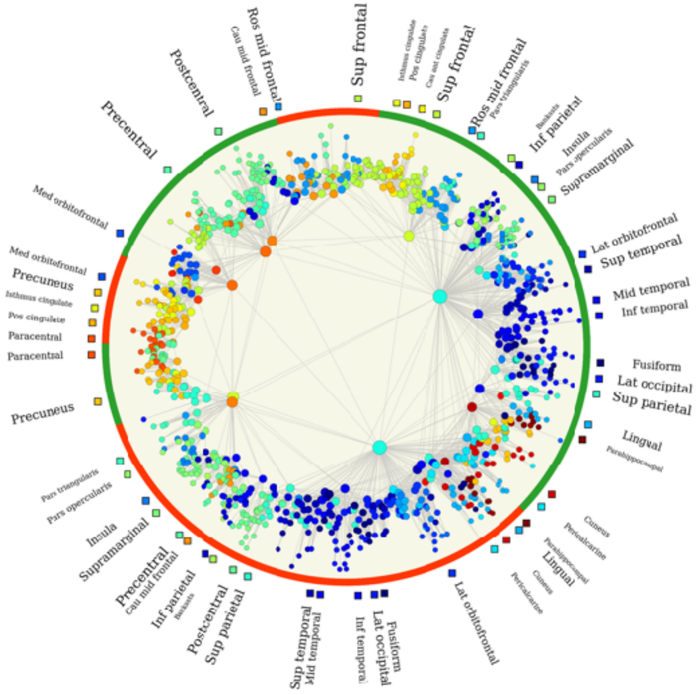
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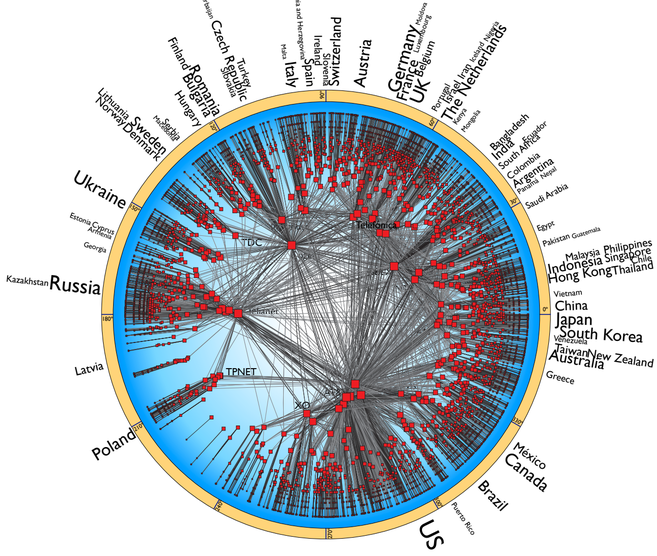
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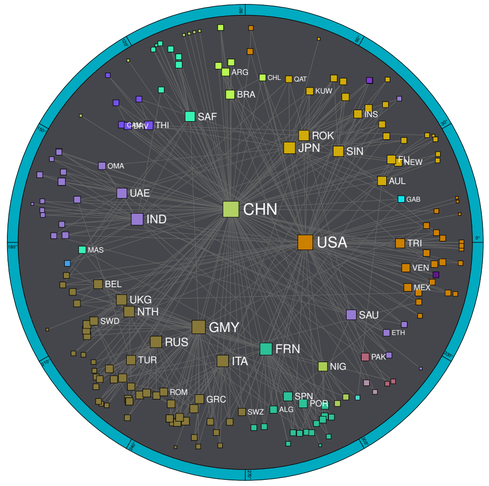
A powerful and versatile framework

- Amenable to many analytical calculations [1,2,3]
- Generalizable to weighted [5], bipartite [6,7,8], multiplex [9,10] and growing [11] networks
- Geometrical interpretation of preferential attachment [11]
- Parsimonious explanation of self-similarity [4]
- Generalizable to networks with community structure [12,13,14]
- Mapping of real complex networks unto hyperbolic space [15,16]
  - Reproduction of additional properties than the ones used to fit the parameters [15].
  - Identification of biochemical pathways in E. Coli [8]
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  - Organization of the human connectome [18,20]
  - Self-similar architecture [19]
  - Evolution of hierarchy in international trade [21]
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[1] Phys. Rev. E 80, 035101 (2009)
[2] Phys. Rev. E 82, 036106 (2010)
[3] Phys. Rev. Research 2, 023040 (2020)
[4] Phys. Rev. Lett. 100, 078701 (2008)
[5] Nat. Commun. 8, 14103 (2017)
[6] Phys. Rev. E 84, 026114 (2011)
[7] Phys. Rev. E 95, 032309 (2017)
[8] Mol. Biosyst. 8, 843 (2012)
[9] Nat. Phys. 12, 1076 (2016)
[10] Phys. Rev. Lett. 118, 218301 (2017)
[11] Nature 489, 537 (2012)
[12] Sci. Rep. 5, 9421 (2015)
[13] J. Stat. Phys. 173, 775 (2018)
[14] New J. Phys. 20, 052002 (2018)
[15] New J. Phys. 21, 123033 (2019)
[16] Nat. Commun. 8, 1615 (2017)
[17] Nat. Commun. 1, 62 (2010)
[18] PNAS 117, 20244 (2020)
[19] Nat. Phys. 14, 583 (2018)
[20] PLOS Comput. Biol. 16, e1007584 (2020)
[21] Sci. Rep. 6, 33441 (2016)

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Review Article | Published: 29 January 2021

Network geometry

Marián Boguñá, Ivan Bonamassa, Manlio De Domenico ✉, Shlomo Havlin, Dmitri Krioukov & M. Ángeles Serrano

Nature Reviews Physics **3**, 114–135 (2021)

[1] Phys. Rev. E 80, 035101 (2009)
[2] Phys. Rev. E 82, 036106 (2010)
[3] Phys. Rev. Research 2, 023040 (2020)
[4] Phys. Rev. Lett. 100, 078701 (2008)
[5] Nat. Commun. 8, 14103 (2017)
[6] Phys. Rev. E 84, 026114 (2011)
[7] Phys. Rev. E 95, 032309 (2017)
[8] Mol. Biosyst. 8, 843 (2012)
[9] Nat. Phys. 12, 1076 (2016)
[10] Phys. Rev. Lett. 118, 218301 (2017)
[11] Nature 489, 537 (2012)
[12] Sci. Rep. 5, 9421 (2015)
[13] J. Stat. Phys. 173, 775 (2018)
[14] New J. Phys. 20, 052002 (2018)
[15] New J. Phys. 21, 123033 (2019)
[16] Nat. Commun. 8, 1615 (2017)
[17] Nat. Commun. 1, 62 (2010)
[18] PNAS 117, 20244 (2020)
[19] Nat. Phys. 14, 583 (2018)
[20] PLOS Comput. Biol. 16, e1007584 (2020)
[21] Sci. Rep. 6, 33441 (2016)

Three challenges in modeling directed networks

Properties of any metric space

Identity of indiscernibles	$d(x,y) = 0 \quad \Leftrightarrow \quad x = y$
Non-negativity	$d(x,y) \geq 0$
Symmetry	$d(x,y) = d(y,x)$
Triangle inequality	$d(x,y) \leq d(x,z) + d(z,y)$

with community structure[29,32,115]. The model has yet to be extended to directed networks, because it is not clear how to reconcile the intrinsic symmetry of metric distances with asymmetric interactions among nodes.

Nat. Rev. Phys. 3, 114 (2021)

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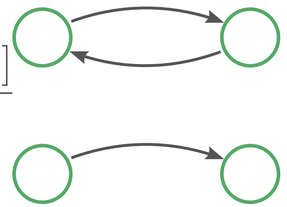
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Reciprocity : cycles of length 2

$$r = \frac{L^{\leftrightarrow}}{L} = \frac{\text{[number of reciprocal links]}}{\text{[number of links]}}$$



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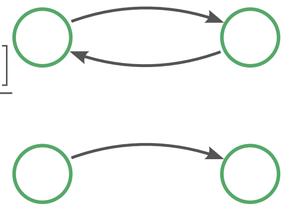
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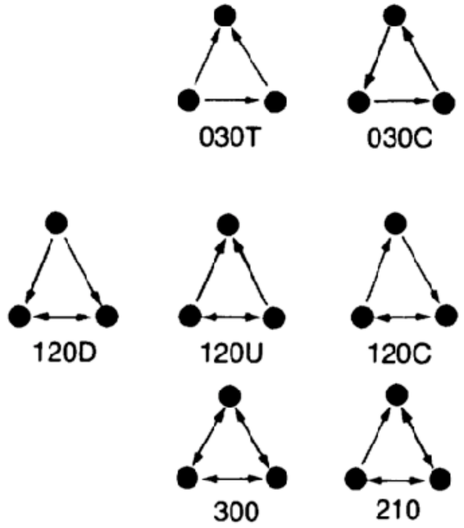
Nat. Rev. Phys. 3, 114 (2021)

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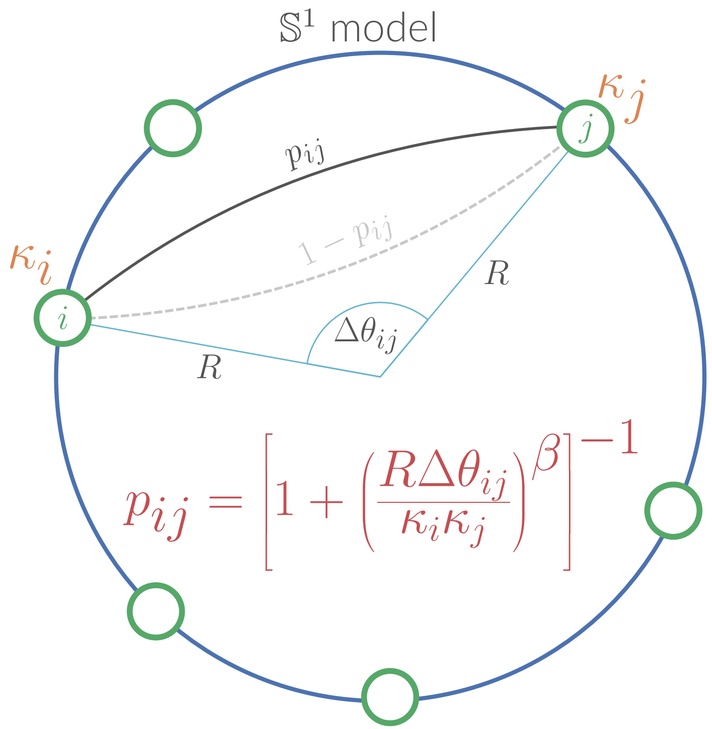


Clustering : 7 cycles of length 3

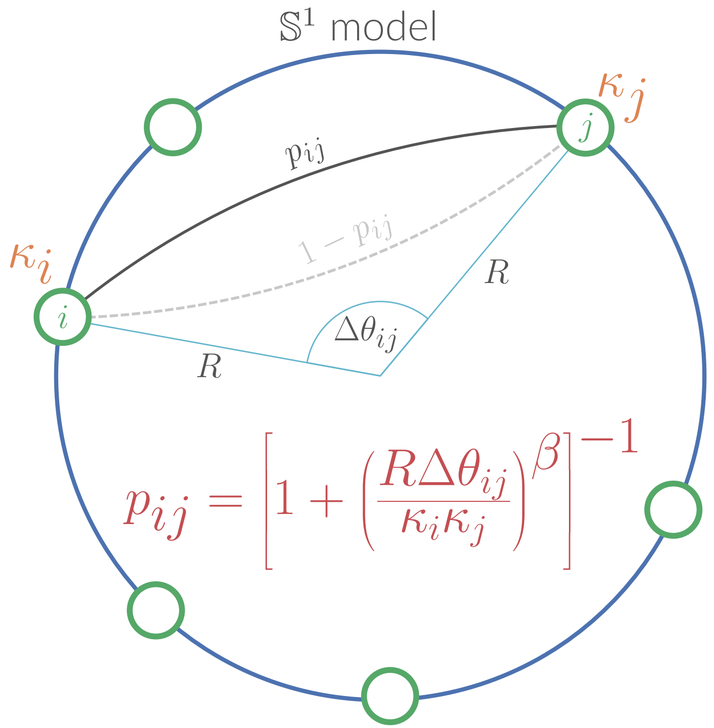


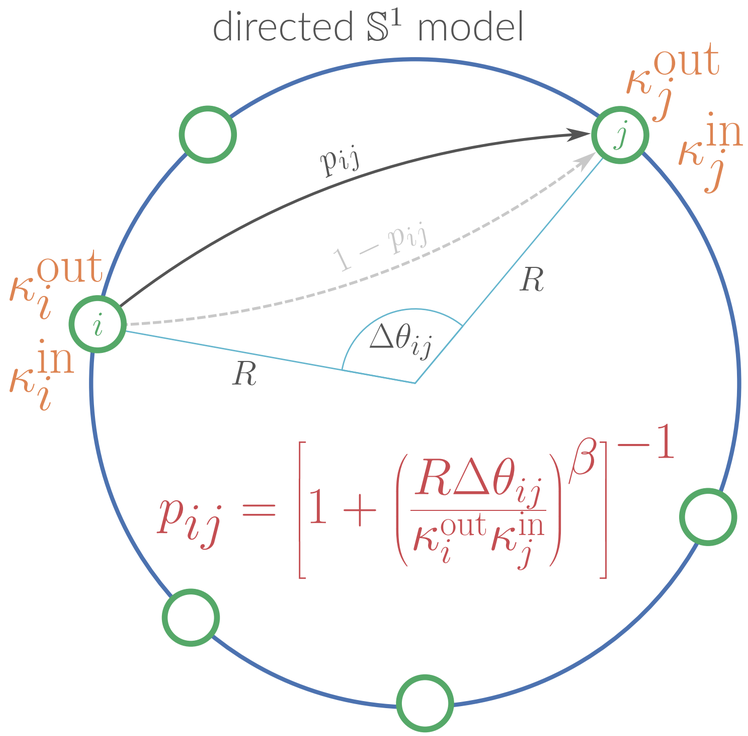
Adapted from Holland & Leinhardt. Local Structure in Social Networks. Sociol. Methodol., 7, 1–45 (1976)

The directed $\mathbb{S}^1$ model: A straightforward generalization



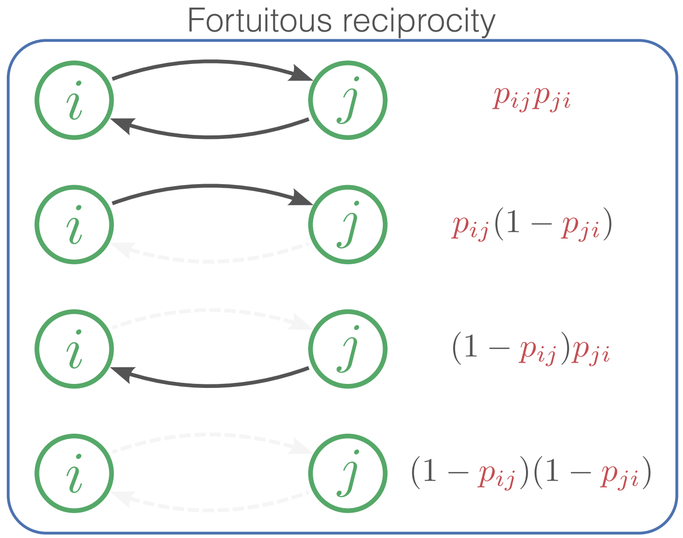
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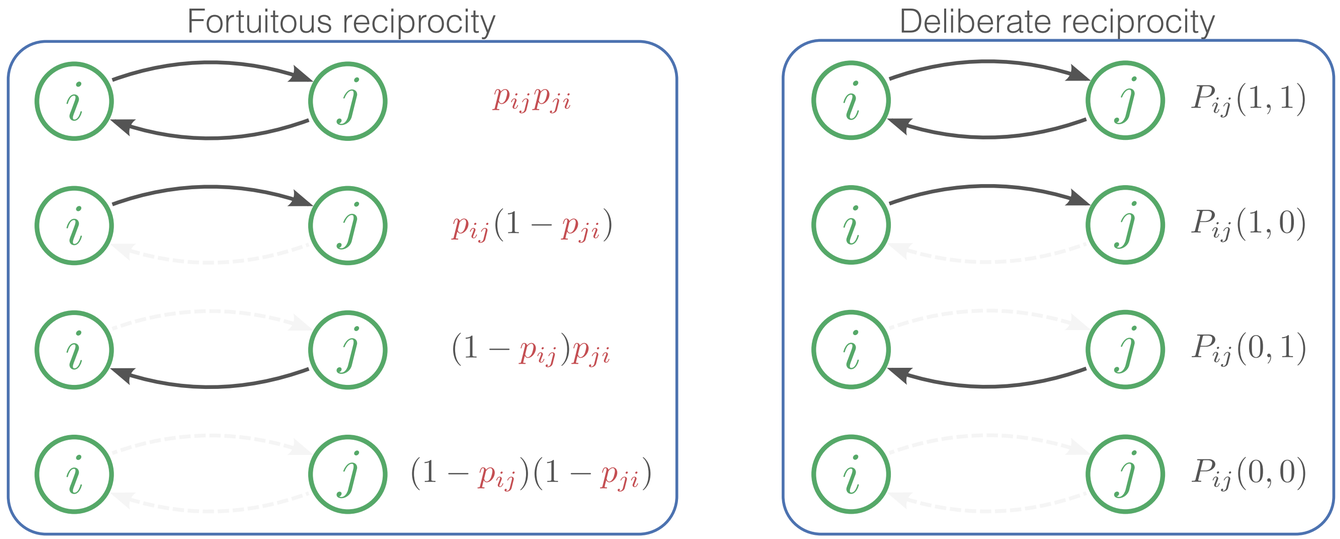
The directed-reciprocal $\mathbb{S}^1$ model: A new link between connections and distances

A random network model defines the probability p_{ij} for a directed link to exist from node i to node j.



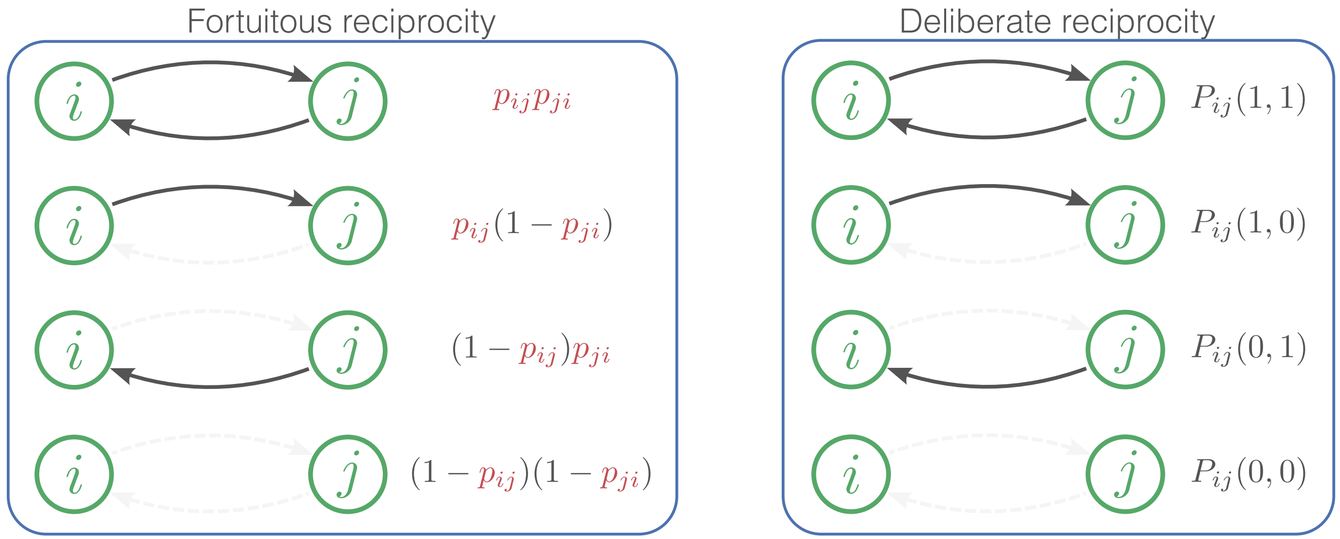
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Level of reciprocity controlled with parameter $-1 \leq \nu \leq 1$

$$P_{ij}(1,1) = \begin{cases} (1-\nu)p_{ij}p_{ji} + \nu \min\{p_{ij}, p_{ji}\} & 0 \leq \nu \leq 1 \\ (1+\nu)p_{ij}p_{ji} + \nu(1-p_{ij}-p_{ji})H(p_{ij}+p_{ji}-1) & -1 \leq \nu \leq 0 \end{cases}$$

$\nu = 1$: maximal reciprocity

$\nu = 0$: fortuitous reciprocity

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Condition 1: Preserves marginal probabilities

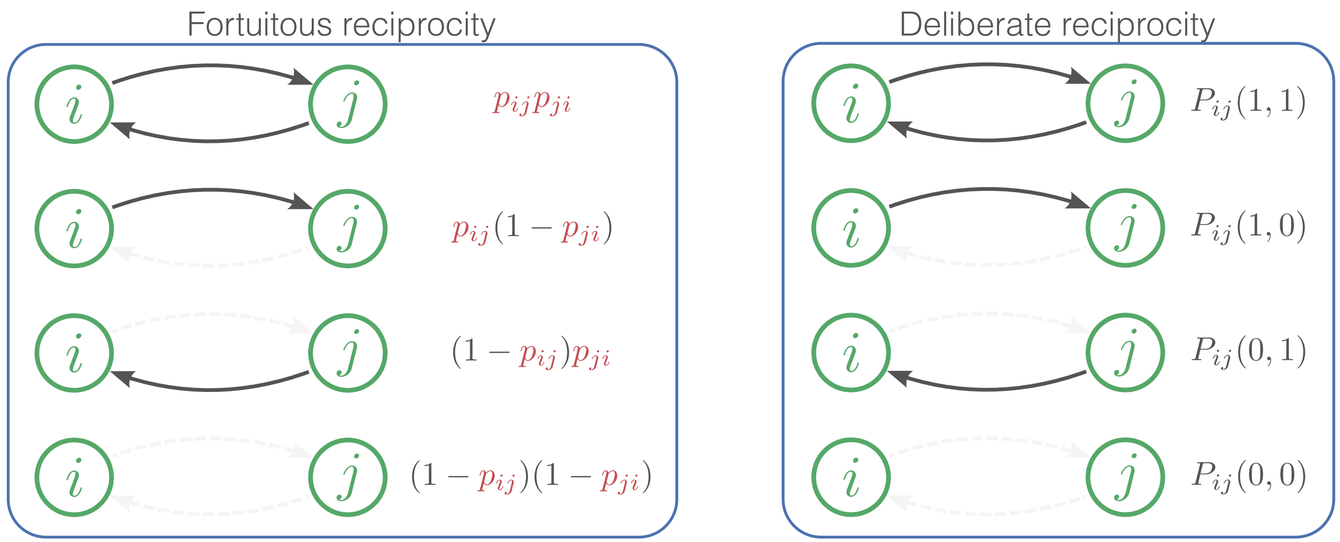
$$P_{ij}(1,0) + P_{ij}(1,1) = p_{ij}$$
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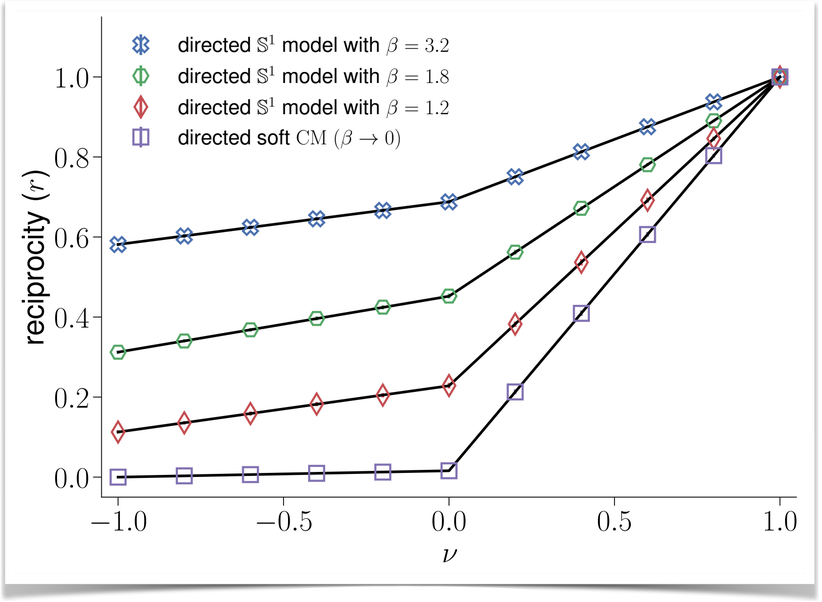
Condition 2: Normalized

$$\sum_{a_{ij}=0}^{1} \sum_{a_{ji}=0}^{1} P_{ij}(a_{ij}, a_{ji}) = 1$$

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Fitting the directed-reciprocal $\mathbb{S}^1$ model to real networks

The likelihood of a network in the directed-reciprocal S^1 model is

$$\mathbb{P}\Big(\overbrace{\mathbb{A}}^{\text{adj. mat.}} \;\Big|\; \underbrace{\{\kappa_i^{\text{in}}\},\{\kappa_i^{\text{out}}\},\{\theta_i\}}_{\text{coordinates}}, \overbrace{\beta,\nu}^{\text{param.}} \Big) = \underbrace{\prod_{i=1}^{N}\prod_{j=i+1}^{N} P_{ij}(a_{ij},a_{ji})}_{\text{likelihood}}$$

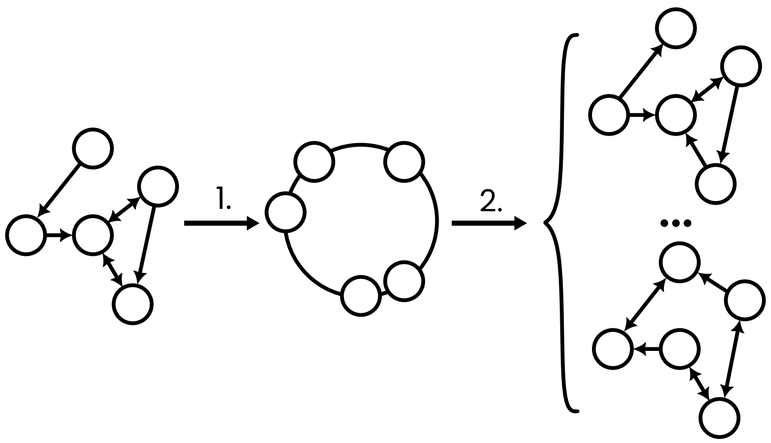
which we maximize using a combination of analytical results and numerical heuristics.

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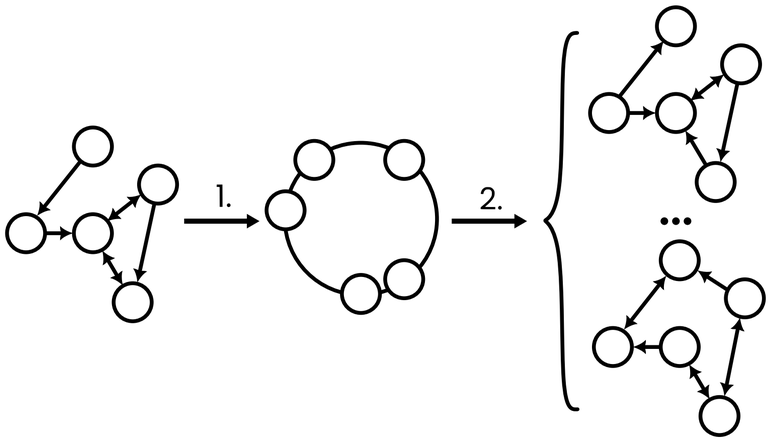


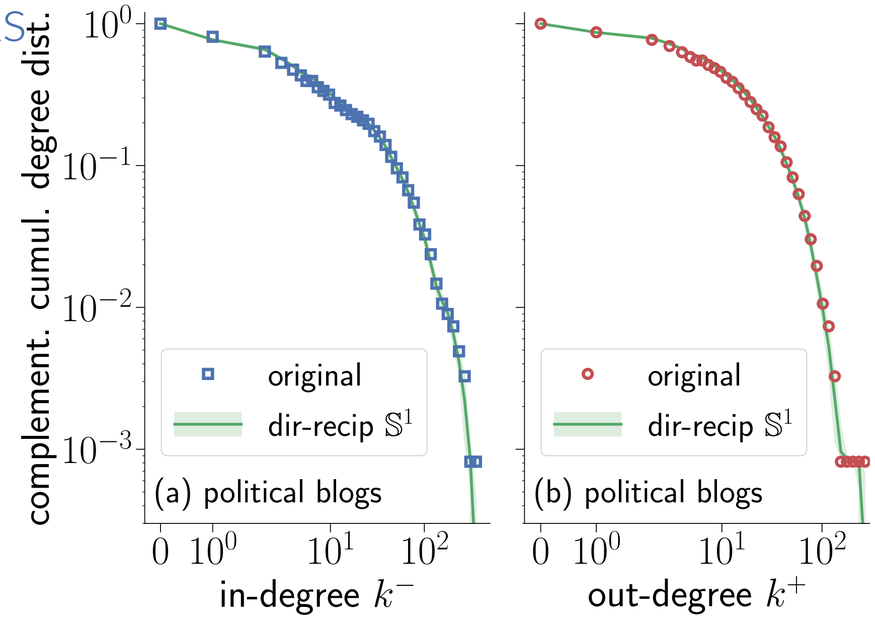
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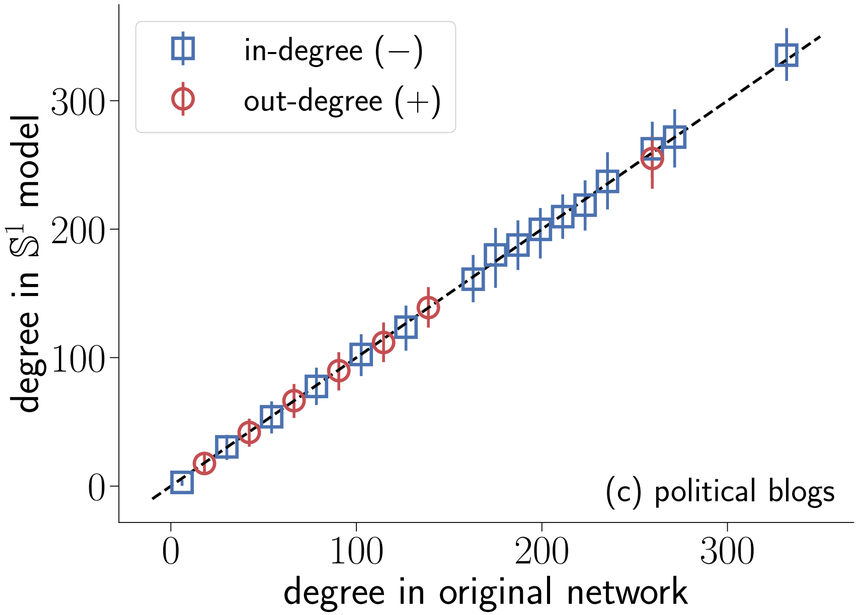
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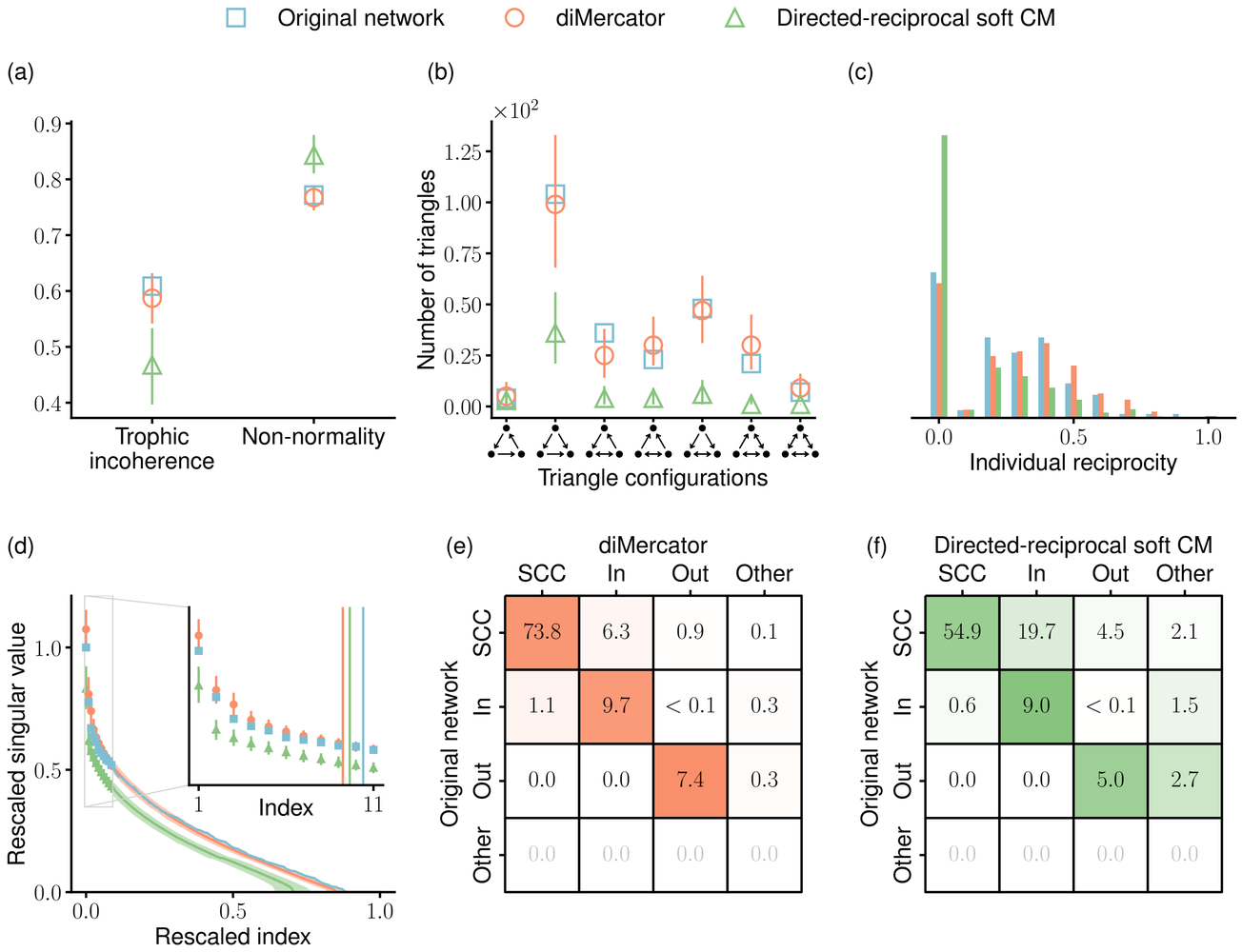
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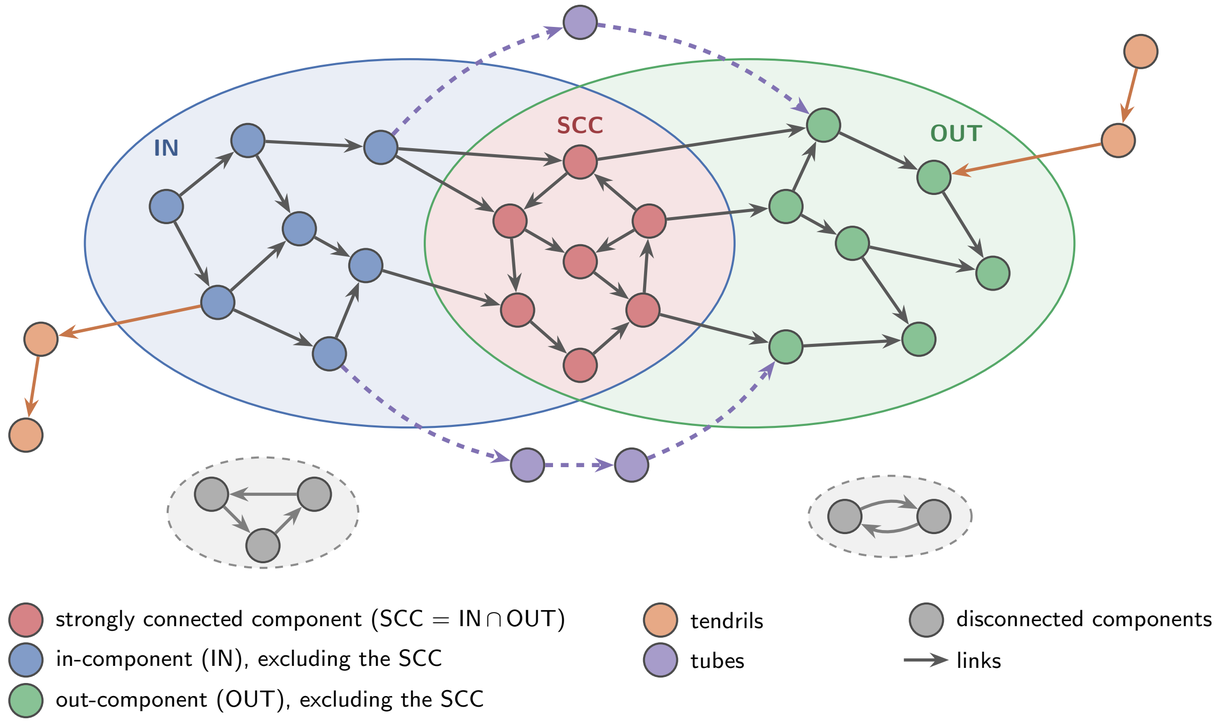




Fitting the directed-reciprocal $\mathbb{S}^1$ model to real networks



Bow-tie structure



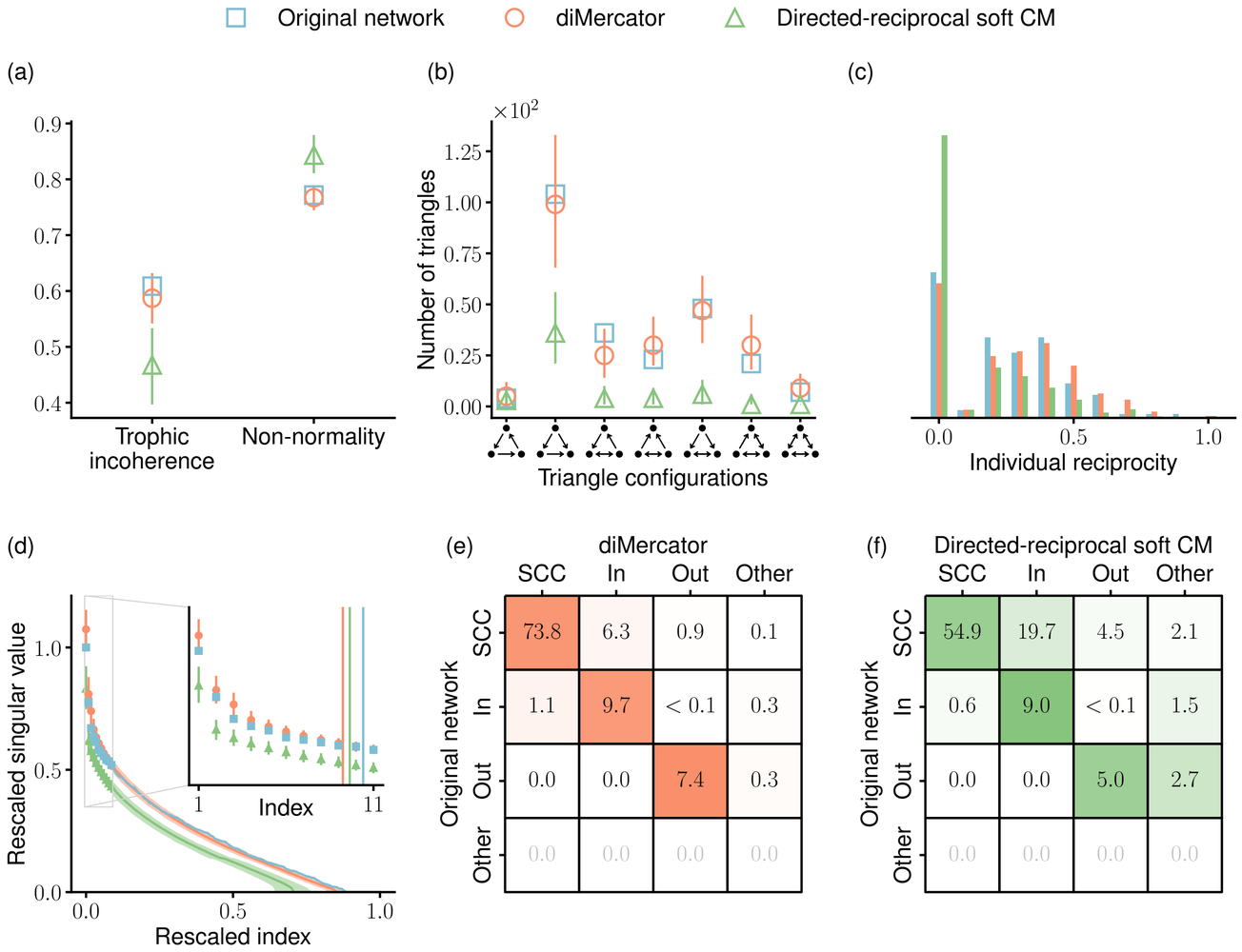
(e) diMercator

Original network \	SCC	In	Out	Other
SCC	73.8	6.3	0.9	0.1
In	1.1	9.7	< 0.1	0.3
Out	0.0	0.0	7.4	0.3
Other	0.0	0.0	0.0	0.0

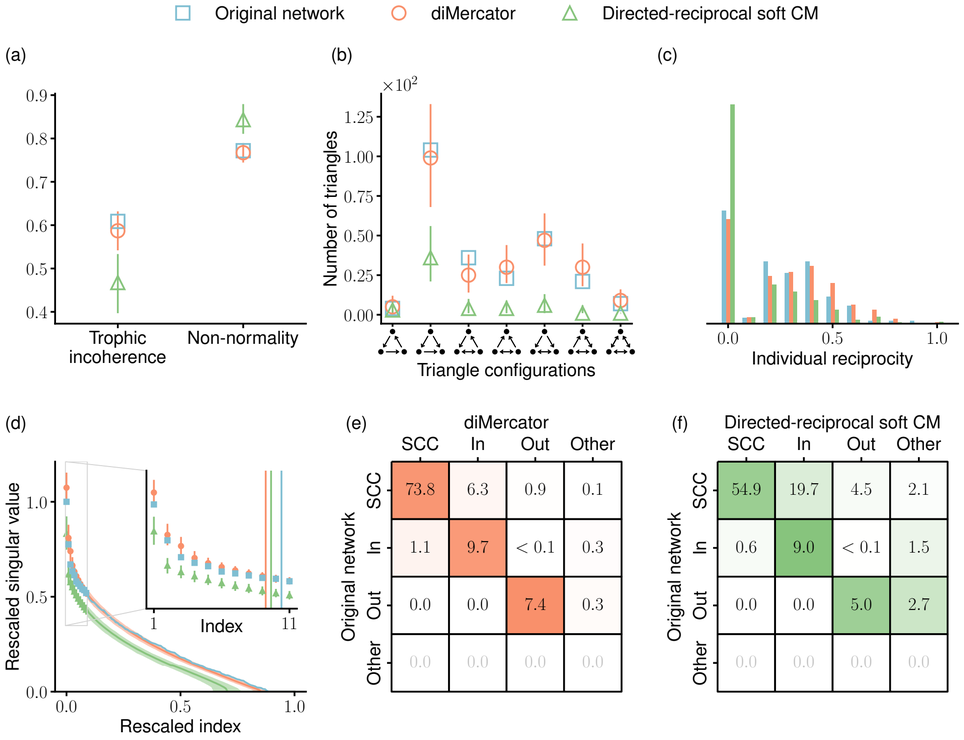
(f) Directed-reciprocal soft CM

Original network \	SCC	In	Out	Other
SCC	54.9	19.7	4.5	2.1
In	0.6	9.0	< 0.1	1.5
Out	0.0	0.0	5.0	2.7
Other	0.0	0.0	0.0	0.0

Fitting the directed-reciprocal $\mathbb{S}^1$ model to real networks



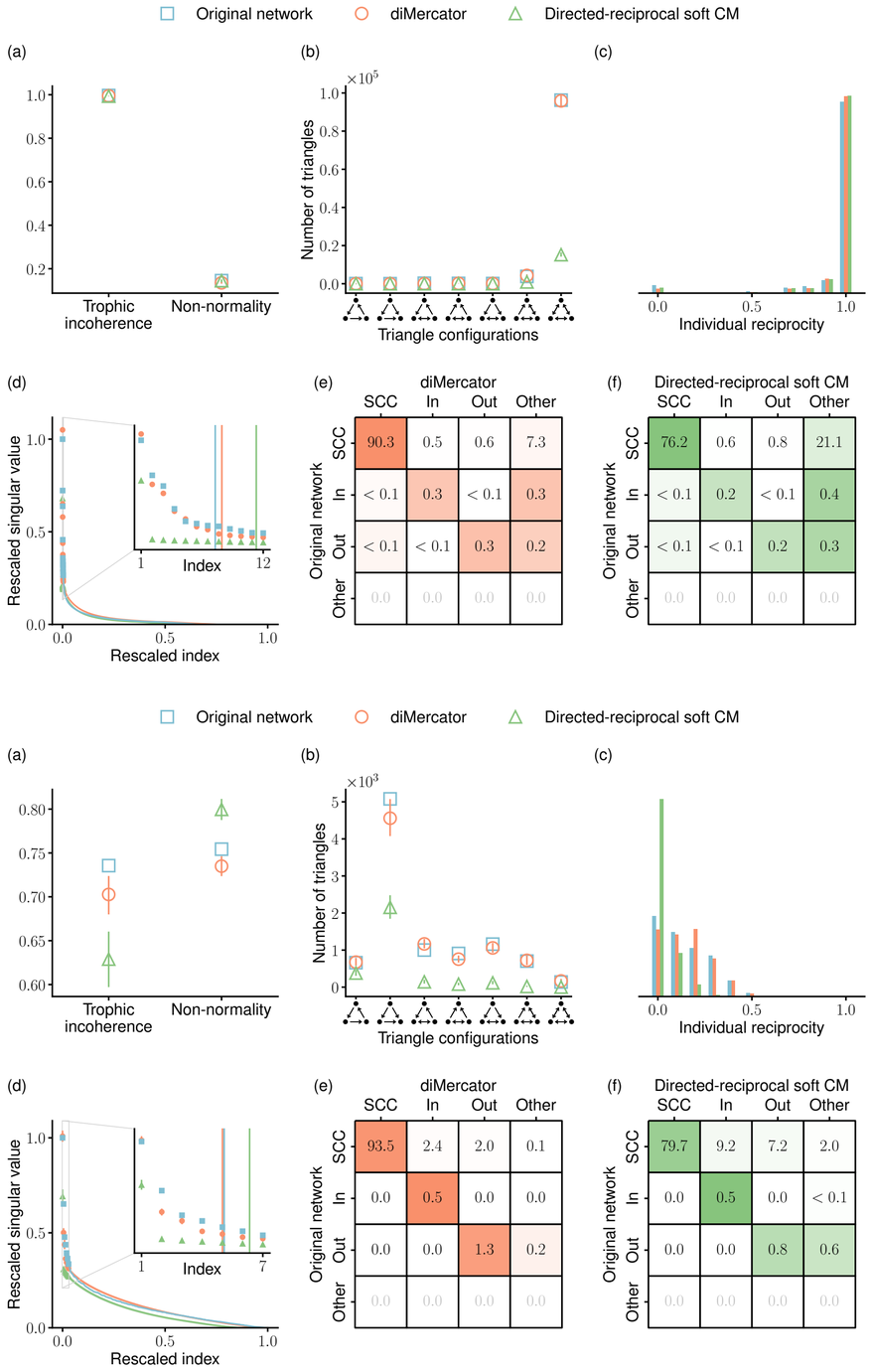
Fitting the directed-reciprocal $\mathbb{S}^1$ model to real networks



Fixing the degree sequence (on average), the global density of triangles and global reciprocity strongly constrains

- the non-normality [1]
- the trophic coherence [2]
- the clustering patterns
- the individual reciprocities
- the singular values
- the bow-tie organization

[1] Sci. Adv. 4, eaau9403 (2018)
[2] PNAS 111, 17923 (2014)

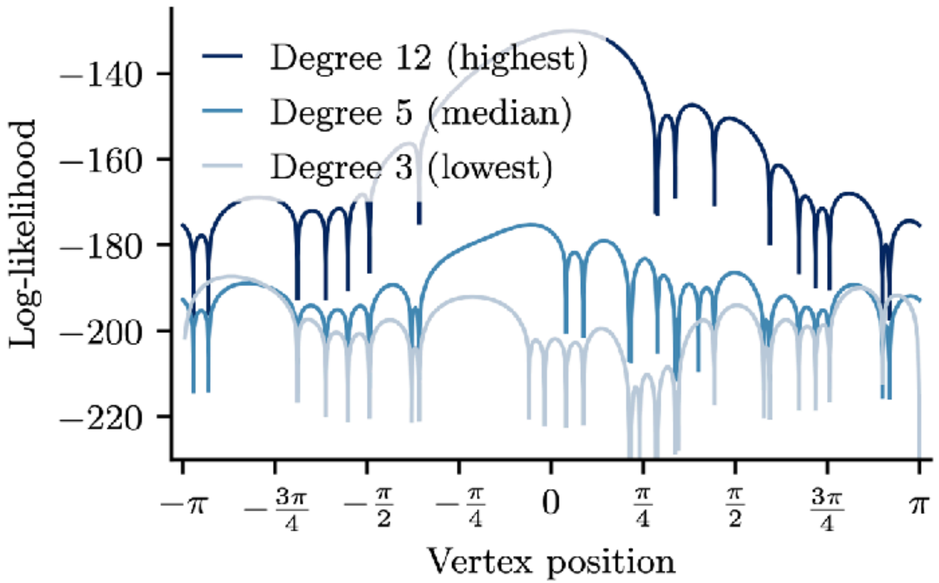


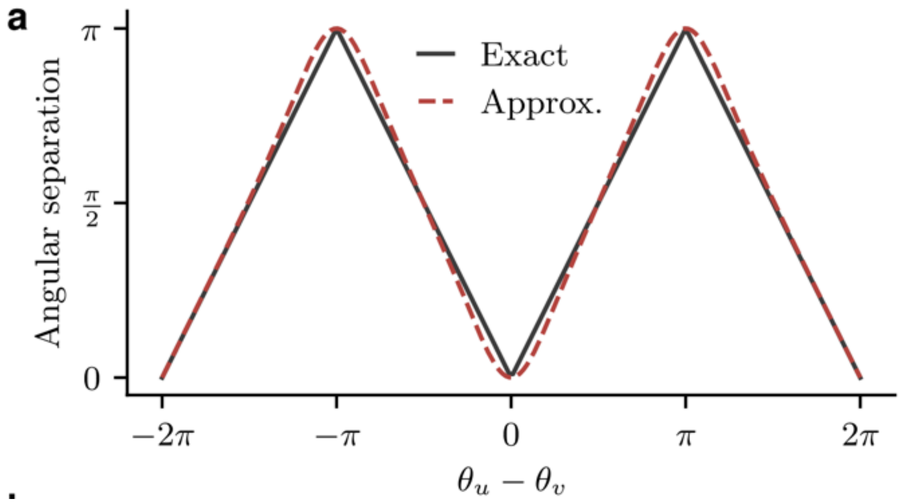
Challenges

Heterogeneous random geometric graph models are prime candidates to model real networked complex systems.

But they rely heavily on our capacity to find high-quality embeddings of the original datasets.

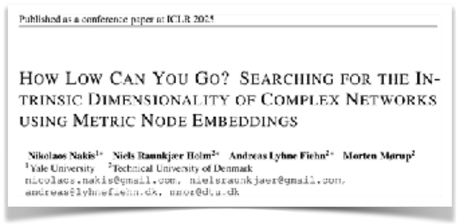
- ▷ Difficult optimization problem
  - rugged landscape
  - numerous symmetries (rotation, reflection, graph automorphisms)
  - gradient not always well defined
- ▷ Out-of-the-box solutions do not work well
  - Hamiltonian Monte Carlo
  - gradient descent
- ▷ Current state-of-the-art embedding methods
  - rely on heuristics
  - do not provide uncertainties (loglikelihood maximization)





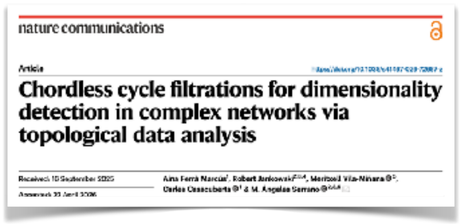
Challenges

What dimension should the embedding space have?


Published as a conference paper at ICLR 2025

HOW LOW CAN YOU GO? SEARCHING FOR THE INTRINSIC DIMENSIONALITY OF COMPLEX NETWORKS USING METRIC NODE EMBEDDINGS

Nikolaos Nakis[1*] Niels Raunkjær Holm[2*] Andreas Lyhne Fiehn[2*] Morten Mørup[2]
[1]Yale University [2]Technical University of Denmark
nikolaos.nakis@gmail.com, nielsraunkjaer@gmail.com,
andreasfiehn@hotmail.dk, mmor@dtu.dk


nature communications

Article

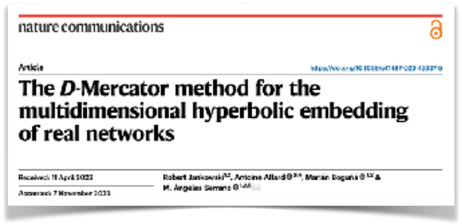
Chordless cycle filtrations for dimensionality detection in complex networks via topological data analysis


PNAS nexus

PNAS Nexus, 2023, 2, 1–12
https://doi.org/10.1093/pnasnexus/pgad136
Advance access publication 18 April 2023
Research Report

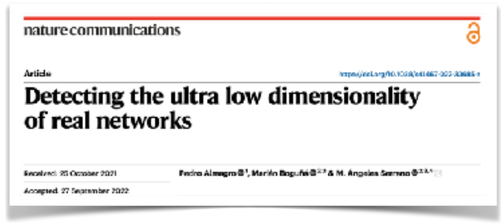
Dimension matters when modeling network communities in hyperbolic spaces

Béatrice Déay[1,2,*], Patrick Desrosiers[3,4,5] and Antoine Allard[3,4,*]


nature communications

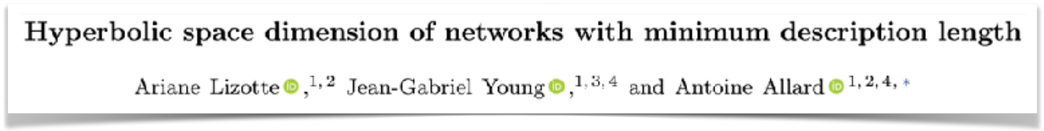
Article

The *D*-Mercator method for the multidimensional hyperbolic embedding of real networks


nature communications

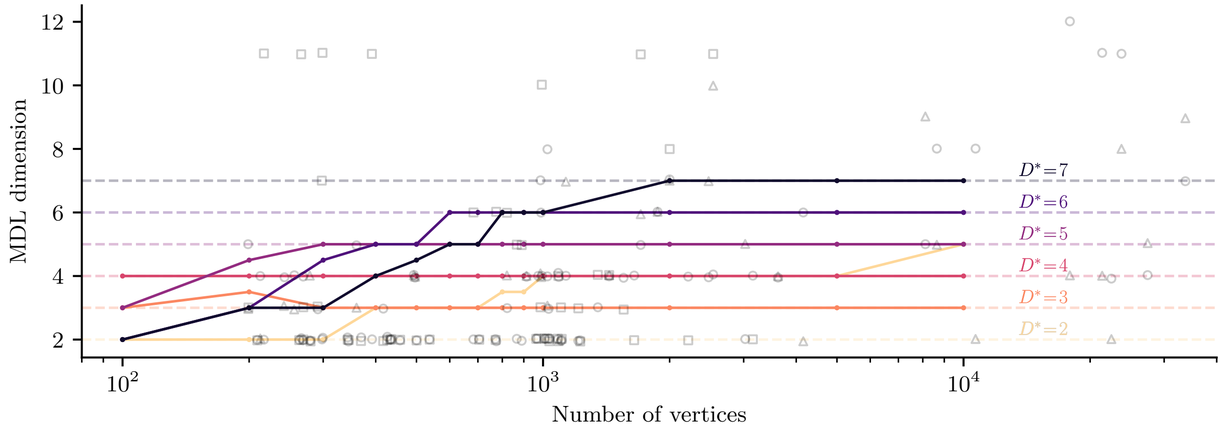
Article

Detecting the ultra low dimensionality of real networks

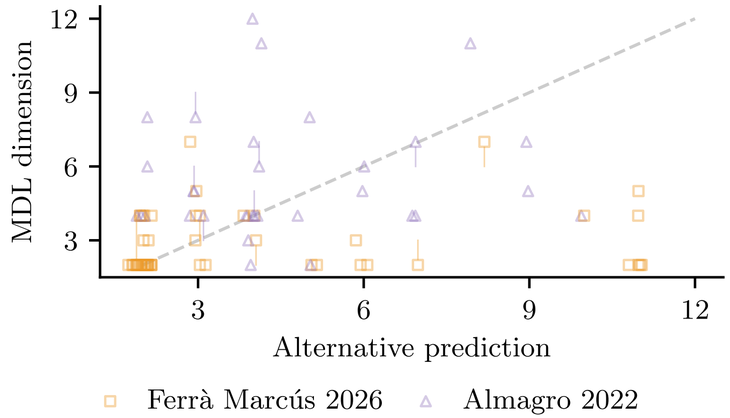

Hyperbolic space dimension of networks with minimum description length

Ariane Lizotte[1,2] Jean-Gabriel Young[1,3,4] and Antoine Allard[1,2,4,*]

Dependence on the number of vertices?



Dependence on the inference method?



Main takeaways

- Rethinking the link between connections and distances allows the extension of the network geometry framework to directed networks.
- Assuming an underlying geometry is a simple modeling approach that can reproduce several complex topological properties observed in real networks.

Challenges

- Network geometry is a promising framework, but its applicability relies heavily on our capacity to find high-quality embeddings of the original datasets, which is hard.
- We cannot power our way through this challenge using GPUs or supercomputers: we need to design smarter algorithms.



M. Ángeles Serrano
Universitat de Barcelona & ICREA



Jérémi Lesage
Université Laval

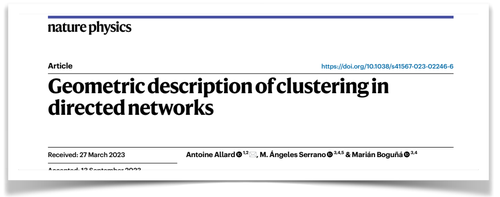
Marián Boguñá
Universitat de Barcelona



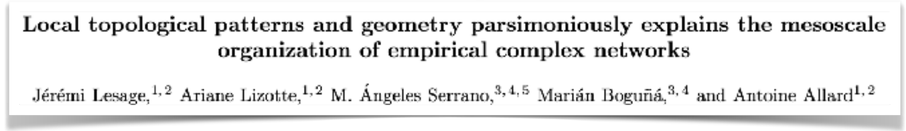


Jean-Gabriel Young
University of Vermont

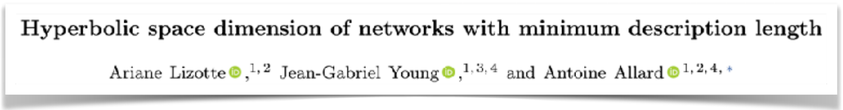
Ariane Lizotte
Université Laval





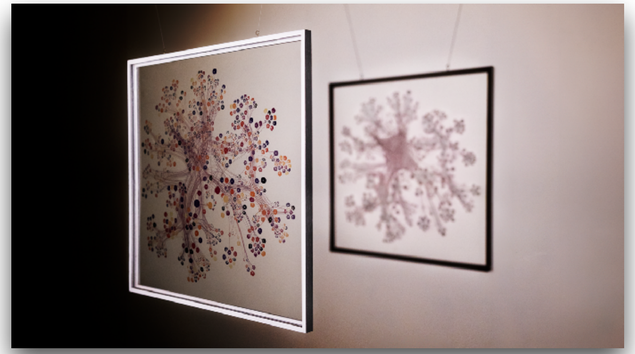


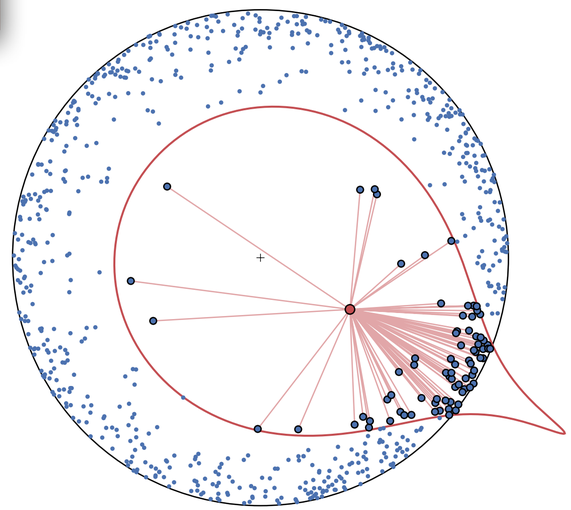
Preprints on arXiv expected fall 2026



Theoretical justifications exist for the use of "simple" models of complex networks and of the dynamics they support.

Simple models can reproduce nontrivial complex connections patterns observed in real networks, but there are still several challenges ahead to bridge the gap between our models and real networks.

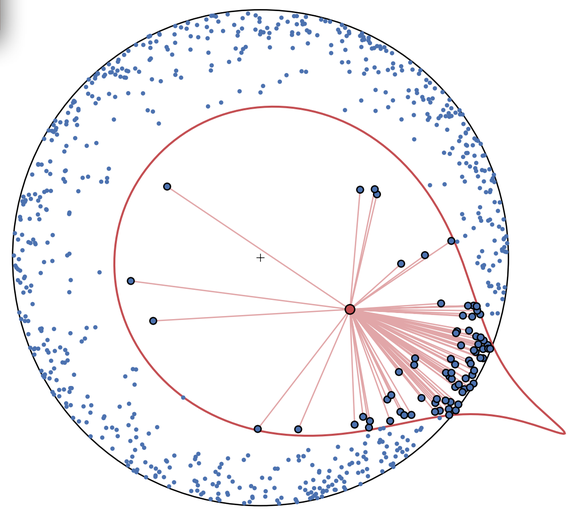




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There's no love in a carbon atom,
No hurricane in a water molecule,
No financial collapse in a dollar bill.
- P. S. Dodds

complexityexplained.github.io

Dynamica

UNIVERSITÉ LAVAL

CIMMUL
Centre interdisciplinaire en modélisation mathématique de l'Université Laval

Alliance de recherche numérique du Canada

NSERC CRSNG

Fonds de recherche Québec